

BALLISTIC TRANSPORT AND ABSOLUTE CONTINUITY OF ONE-FREQUENCY SCHRÖDINGER OPERATORS

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ABSTRACT. For the solution $u(t)$ to the discrete Schrödinger equation

$$i \frac{d}{dt} u_n(t) = -(u_{n+1}(t) + u_{n-1}(t)) + V(\theta + n\alpha)u_n(t), \quad n \in \mathbb{Z},$$

with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $V \in C^\omega(\mathbb{T}, \mathbb{R})$, we consider the growth rate with t of its diffusion norm $\langle u(t) \rangle_p := (\sum_{n \in \mathbb{Z}} (n^p + 1)|u_n(t)|^2)^{\frac{1}{2}}$, and the (non-averaged) transport exponents

$$\beta_u^+(p) := \limsup_{t \rightarrow \infty} \frac{2 \log \langle u(t) \rangle_p}{p \log t}, \quad \beta_u^-(p) := \liminf_{t \rightarrow \infty} \frac{2 \log \langle u(t) \rangle_p}{p \log t}.$$

We will show that, if the corresponding Schrödinger operator has purely absolutely continuous spectrum, then $\beta_u^\pm(p) = 1$, provided that $u(0)$ is well localized.

1. INTRODUCTION AND MAIN RESULT

For the discrete quasi-periodic Schrödinger operator

$$(L_\theta u)_n = -(u_{n+1} + u_{n-1}) + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z},$$

with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the frequency and V the potential function on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, we consider the dynamics of the equation

$$(1.1) \quad i \frac{d}{dt} u(t) = L_\theta u(t).$$

For its solution $u(t)$, we want to observe the growth rate with t of the “diffusion norm”(or in some context also known as “2nd-moment of the position”):

$$\langle u(t) \rangle_2 := \left(\sum_{n \in \mathbb{Z}} (n^2 + 1) |u_n(t)|^2 \right)^{\frac{1}{2}}.$$

More generally, we can define the p^{th} -moment of $u(t)$ for any $p \geq 0$ by

$$\langle u(t) \rangle_p := \left(\sum_{n \in \mathbb{Z}} (|n|^p + 1) |u_n(t)|^2 \right)^{\frac{1}{2}}.$$

We define the subspace $\mathcal{W}^p(\mathbb{Z}) := \{u \in \ell^2(\mathbb{Z}) : \langle u \rangle_p < \infty\}$.

It is known that when the initial condition $u(0)$ is well-localised, we have $\langle u(t) \rangle_p < \infty$ for any finite t and $p \geq 0$ (see Theorem 2.22 of [14]). One standard way to describe the propagation of $u(t)$ in space is to consider the asymptotic growth of the p^{th} -moment norm. This is stated in terms of transport exponents

$$\beta_u^+(p) := \limsup_{t \rightarrow \infty} \frac{2 \log \langle u(t) \rangle_p}{p \log t}, \quad \beta_u^-(p) := \liminf_{t \rightarrow \infty} \frac{2 \log \langle u(t) \rangle_p}{p \log t}.$$

There is an extensive study of transport exponents and the time-averaged variants. It is understood that the transport behaviour is intimately related to the spectral properties of the operator. In the case where L_θ has only pure point spectrum, Simon[29] showed that for compactly supported $u(0)$,

$$\lim_{t \rightarrow \infty} t^{-1} \langle u(t) \rangle_2 = 0.$$

On the other hand, the absolutely continuous(from now on, a.c. for short) part of the spectrum is expected to be associated with the strongest transport property, which is usually called ballistic transport. More precisely, for the solution $u(t)$ to Eq. (1.1), it is expected that the norm $\langle u(t) \rangle_p$ grows like $t^{p/2}$. As a general result in this direction, there is a time-averaged statement by Guarneri-Combes-Last theorem[24], which shows that, in the presence of a.c. spectrum, for any well-localised $u \neq 0$ in the subspace of $\ell^2(\mathbb{Z})$ corresponding to the a.c. spectrum, there exists some positive constant C such that

$$\liminf_{t \rightarrow \infty} \frac{1}{T} \int_0^T \langle u(t) \rangle_p^2 dt \geq CT^p.$$

There is also a large body of literature devoted to the study of transport property for operators with singular continuous spectrum. We will refer the readers to survey [14] and introductions in [13] for details. For more descriptions of the diffusion norm of nonlinear operators, we refer to [9].

The main subject of this paper is to investigate the transport property of operators with a.c. spectrum. It is fairly natural to ask whether one can go beyond the averaged version in Guarneri-Combes-Last theorem. It turns out that this is not a simple generalisation, possibly due to lack of good spectral quantity associated to terms like $\langle u(t) \rangle_p$. Recently, Damanik-Lukic-Yessen[12] have shown the stronger version of ballistic motion for $p = 2$ (i.e., the above inequality without time-averaging) for the periodic Schrödinger equation. This is an extension of the work of Asch-Knauf[2] for Schrödinger operators. Zhao[32] has proven it for the quasi-periodic ones with small potential and Diophantine frequencies. Both results correspond to the Schrödinger operator with purely a.c. spectrum.

In this paper, we establish a complete link between purely a.c. spectrum and ballistic transport for (analytic) one-frequency Schrödinger operator. The main result is stated as follows.

Theorem 1. *Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $V \in C^\omega(\mathbb{T}, \mathbb{R})$ such that the Schrödinger operator $L = L_\theta$ has purely a.c. spectrum for a.e. θ . Given any $\eta > 0$ and $p \geq 0$.*

- 1) Consider the solution $u(t)$ of Eq. (1.1) with $u(0) \in \mathcal{W}^{p'}(\mathbb{Z}) \setminus \{0\}$, where $p' = p$ if $p \in 2\mathbb{Z}$ and $p' = 2\lfloor \frac{p}{2} \rfloor + 2$ otherwise. For every $\theta \in \mathbb{T}$, we have

$$\lim_{t \rightarrow \infty} \frac{\langle u(t) \rangle_p^2}{t^{p+\eta}} = 0.$$

- 2) If $p \geq 2$ or $p = 0$ and $u(0) \in \mathcal{W}^p(\mathbb{Z}) \setminus \{0\}$, or if $0 < p < 2$ and $u(0) \in \mathcal{W}^4(\mathbb{Z}) \setminus \{0\}$, then for a.e. $\theta \in \mathbb{T}$, we have

$$\lim_{t \rightarrow \infty} \frac{\langle u(t) \rangle_p^2}{t^{p-\eta}} = \infty.$$

In particular, for a.e. $\theta \in \mathbb{T}$, for any exponentially decaying $u(0) \neq 0$, we have $\beta_u^+(p) = \beta_u^-(p) = 1$.

REMARK 1. In view of the work of Damanik-Lukic-Yessen for the periodic potential case (Theorem 1.6 of [12]), the conclusion of Theorem 1 holds for any $\alpha \in \mathbb{R}$.

REMARK 2. According to Corollary 1.7 of [4], the conclusion of 1 holds if V is close to constant.

Besides the intrinsic interest in this problem, another motivation can be found in connection with the so-called XY spin chain, studied in many-body quantum physics. In [12], the authors established lower bound for the Lieb-Robinson velocity for the anisotropic XY chain on \mathbb{Z} with periodic parameters as an application of their proof of the ballistic motions. Also through this connection with Schrödinger operators, Kachkovskiy[22] has proven similar lower bound for a class of isotropic quasi-periodic XY spin, corresponding to analytic cocycles which are reducible for almost every energy, including the small analytic regime in [32], and the cases with purely a.c. spectrum and a single Diophantine frequency. It is explained in [22] that although the transport property in two models share some connections, the averaged lower bound of Guarneri-Combes-Last does not translate into useful informations for XY spin chains. We hope that the method in this paper could provide some information on how to approach XY spin chains.

Idea of Proof. Since the general ballistic upper bound is known (see Theorem 3 in Section 5), we only need to show the lower bound.

Following the main strategy of [32], we relate the growth of the diffusion norm to the so-called “modified spectral transformation”. Roughly speaking, one comes down to show that the Bloch-wave at different sites has decaying correlation with respect to some well-chosen measure. In [32], a natural candidate measure for this construction is the measure defined by the integrated density of states. The derivative of the Floquet exponent serves as the source of the decay. While in our case, two difficulties arises.

- The first is that the usual KAM breaks down, namely in general one can no longer reduce the Schrödinger cocycle to constant. Avila-Fayad-Krikorian[6] developed a theory which allows one to reduce the cocycle to a (phase-dependent) rotation. Through this type of reducibility, we can construct the “generalized Bloch-wave” with “phase-dependent Floquet exponent”. The phase-dependence complicated the matter (see Subsection 5.4).
- The second difficulty is that we need to exploit the second order derivatives of the modified spectral measure, while the rotation number only has good first order derivative estimates. This difficulty does not appear in the construction of [32], in which the Floquet exponent is just the rotation number and the form of Bloch-wave is much simpler, due to the reducibility to constant. Then an integration by parts can be performed before exploiting the regularity of the rotation number. While in our case we can not expect to find a parametrisation to accommodate the variation of phase-dependent Floquet exponent at different sites. So we need to

make a non-canonical choice of measure that comes in the construction of the modified spectral transformation. To retain the decay of the correlations, one has to study the regularity of the measure in question. Since one often expects the spectrum to be a Cantor set, we can not expect to choose a measure with summable sequence of Fourier coefficients. This is why we are only able to study the transport exponents but not the linear lower bound as in [32].

To get the ballistic lower exponent, our main observation is: by the renormalization theory developed by Avila-Krikorian([7] and its generalisation [8]), we can initialize the KAM for arbitrarily fine data, which allows us to delay the occurrence of resonance at will. We will construct a measure supported on spectrum adapted to the KAM scheme, with good regularity at finite yet arbitrarily long intervals of scales. Combining with a truncated version of modified spectral transform, we can complete the proof.

2. PRELIMINARIES AND NOTATIONS

2.1. Schrödinger operator and Schrödinger cocycle. We recall some basic notions and well-known results for the quasi-periodic Schrödinger operator $L = L_\theta : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$,

$$(Lu)_n = -(u_{n+1} + u_{n-1}) + V(\theta + n\alpha)u_n,$$

with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $V \in C^\omega(\mathbb{T}, \mathbb{R})$ and the corresponding Schrödinger cocycle $(\alpha, A_{(E,V)})$:

$$(2.1) \quad \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{(E,V)}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \text{ with } A_{(E,V)}(\theta) := \begin{pmatrix} -E + V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $(\alpha, A_{(E,V)})$ is equivalent to the eigenvalue problem $Lu = Eu$.

2.1.1. Spectral measure and integrated density of states. Let $\sigma(L)$ denote the spectrum of L . Fixing any phase $\theta \in \mathbb{T}$ and any $\psi \in \ell^2(\mathbb{Z})$, let $\mu_\theta = \mu_{\theta,\psi}$ be the spectral measure of $L = L_\theta$ corresponding to ψ , which is defined so that

$$\langle (L_\theta - E)^{-1} \psi, \psi \rangle = \int_{\mathbb{R}} \frac{1}{E - E'} d\mu_{\theta,\psi}(E'), \quad \forall E \in \mathbb{C} \setminus \sigma(L).$$

From now on, we restrict our consideration to $\mu_\theta = \mu_{\theta,\delta_{-1}} + \mu_{\theta,\delta_0}$ and just call it the **spectral measure**, where $\{\delta_n\}_{n \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. Since $\{\delta_{-1}, \delta_0\}$ forms a generating basis of $\ell^2(\mathbb{Z})$ [10], that is, there is no proper subset of $\ell^2(\mathbb{Z})$ which is invariant by L and contains $\{\delta_{-1}, \delta_0\}$. In particular the support of μ_θ is $\sigma(L)$ and if μ_θ is a.c. then any $\mu_{\theta,\psi}$, $\psi \in \ell^2(\mathbb{Z})$, is a.c. .

The **integrated density of states** is the function $k : \mathbb{R} \rightarrow [0, 1]$ such that

$$k(E) = \int_{\mathbb{T}} \mu_\theta(-\infty, E] d\theta,$$

which is a continuous non-decreasing surjective function.

2.1.2. Rotation number and Lyapunov exponent. Related to the Schrödinger cocycle, a unique representation can be given for the rotation number $\rho = \rho_{(\alpha, A_{(E,V)})}$. Indeed, the rotation number is defined for more general quasi-periodic cocycles. It is introduced originally by Herman[19] in this discrete case(see also Delyon-Souillard[15], Johnson-Moser[21], Krikorian[23]). For the precise definition, we follow the same presentation as in [6].

Given $A(\cdot) : \mathbb{T} \rightarrow SL(2, \mathbb{R})$, continuous and homotopic to the identity, the same is true for

$$\begin{aligned} F : \mathbb{T} \times \mathbb{S}^1 &\rightarrow \mathbb{T} \times \mathbb{S}^1 \\ (\theta, v) &\mapsto \left(\theta + \alpha, \frac{A(\theta)v}{\|A(\theta)v\|} \right). \end{aligned}$$

Therefore, F admits a continuous lift $\tilde{F} : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$ of the form $\tilde{F}(\theta, x) = (\theta + \alpha, x + f(\theta, x))$ such that $f(\theta, x + 1) = f(\theta, x)$ and $\Pi(x + f(\theta, x)) = \frac{A(\theta)\Pi(x)}{\|A(\theta)\Pi(x)\|}$, where $\Pi : \mathbb{R} \rightarrow S^1$, $\Pi(x) = e^{i2\pi x} := (\cos 2\pi x, \sin 2\pi x)$. In order to simplify the terminology, we can say that \tilde{F} is a lift for (α, A) . The map f is independent of the choice of the lift up to the addition of a constant integer $p \in \mathbb{Z}$. Following [19] and [21], we define the limit

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{F}^k(\theta, x)),$$

which is independent of (θ, x) and where the convergence is uniform in (θ, x) . The class of this number in \mathbb{T} , which is independent of the chosen lift, is called the **fibered rotation number** of (α, A) and denoted by $\rho_{(\alpha, A)}$. Moreover, $\rho_{(\alpha, A)}$ is continuous as a function of A (with respect to the uniform topology on $C^0(\mathbb{T}, SL(2, \mathbb{R}))$, naturally restricted to the subset of A homotopic to the identity).

For the quasi-periodic cocycle $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the **Lyapunov exponent** $\gamma = \gamma_{(\alpha, A)}$ is defined by

$$\gamma_{(\alpha, A)} := \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{T}} \ln |A(\theta + n\alpha) \cdots A(\theta + \alpha)| d\theta.$$

By Kingman's subadditive ergodic theorem,

$$\gamma_{(\alpha, A)} := \lim_{n \rightarrow \infty} \frac{1}{n} \ln |A(\theta + n\omega) \cdots A(\theta + \omega)|.$$

In particular, for quasi-periodic Schrödinger cocycle $(\alpha, A_{(E, V)})$ given in (2.1), a well-known result of Kotani theory shows, if the linear Schrödinger operator L has purely absolutely continuous spectrum, then $\gamma(E) = 0$ a.e. on $\sigma(L)$. Moreover, the Thouless formula relates the Lyapunov exponent to the integrated density of states:

$$\gamma(E) = \gamma_{(\alpha, A_{(E, V)})} = \int_{\mathbb{R}} \ln |E' - E| dk(E').$$

There is also a relation between the rotation number and the integrated density of states:

$$k(E) = \begin{cases} 0, & E \leq \inf \sigma(L) \\ \frac{\rho(E)}{\pi}, & \inf \sigma(L) < E < \sup \sigma(L) \\ 1, & E \geq \sup \sigma(L) \end{cases}.$$

By gap-labelling theorem (see, e.g., [15, 21]), $k(E) = \frac{\rho(E)}{\pi}$ is constant in a gap of $\sigma(L)$ (i.e., an interval in the resolvent set of L), and each gap is labelled with $l \in \mathbb{Z}$ such that $\rho = \frac{l\alpha}{2} \bmod \pi$ in this gap.

2.1.3. The m -functions. The spectral measure $\mu = \mu_\theta$ can be studied through its Borel transform $M = M_\theta$:

$$M(z) = \int \frac{1}{E' - z} d\mu(E').$$

It maps the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ into itself.

From the limit-point theory, for $z \in \mathbb{H}$, there are two solutions u^\pm , with $u_0^\pm \neq 0$, which are ℓ^2 at $\pm\infty$ and satisfying $Lu^\pm = zu^\pm$, defined up to normalization. Let $m^\pm := -\frac{u_{\pm 1}^\pm}{u_0^\pm}$. m^+ and m^- are

Herglotz functions, i.e., they map \mathbb{H} holomorphically into itself (see, e.g., [28] for more properties of Herglotz function). Moreover, it is well known that

$$M = \frac{m^+ m^- - 1}{m^+ + m^-}.$$

By the property of Herglotz function, we know that for almost every $E \in \mathbb{R}$, the non-tangential limits $\lim_{\epsilon \rightarrow 0} m^\pm(E + i\epsilon)$ exist, and they define measurable functions on \mathbb{R} which we still denote $m^\pm(E)$.

We have the following key result of Kotani Theory [28].

LEMMA 1 (Theorem 2.2 of [3]). *For every θ , for a.e. E such that $\gamma(E) = 0$, we have $m^+(E) = m^-(E)$.*

2.1.4. *Classical spectral transformation.* Let $u(E)$ and $v(E)$ be the solutions of the eigenvalue problem

$$Lq = Eq \text{ such that } \begin{pmatrix} u_1 & v_1 \\ u_0 & v_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \text{ We have}$$

Theorem 2 (Chapter 9 of [11]). *There exists a non-decreasing Hermitian matrix $\mu = (\mu_{jk})_{j,k=1,2}$ whose elements are of bounded variation on every finite interval on \mathbb{R} , satisfying*

$$\mu_{jk}(E_2) - \mu_{jk}(E_1) = \lim_{\epsilon \rightarrow 0+} \frac{1}{\pi} \int_{E_1}^{E_2} \Im M_{jk}(v + i\epsilon) dv,$$

at points of continuity E_1, E_2 of μ_{jk} , where on \mathbb{H} ,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} := -\frac{1}{m^+ + m^-} \begin{pmatrix} 1 & m^+ \\ -m^- & -m^+ m^- \end{pmatrix},$$

such that for any $q \in \ell^2(\mathbb{Z})$, with $(g_1(E), g_2(E)) := (\sum_{n \in \mathbb{Z}} q_n u_n(E), \sum_{n \in \mathbb{Z}} q_n v_n(E))$, we have Parseval's equality

$$\sum_{n \in \mathbb{Z}} |q_n|^2 = \int_{\mathbb{R}} \sum_{j,k=1}^2 \bar{g}_j(E) g_k(E) d\mu_{jk}(E).$$

Given any matrix of measures on \mathbb{R} $d\varphi = \begin{pmatrix} d\varphi_{11} & d\varphi_{12} \\ d\varphi_{21} & d\varphi_{22} \end{pmatrix}$, let $\mathcal{L}^2(d\varphi)$ be the space of vectors $G = (g_j)_{j=1,2}$, with g_j functions of $E \in \mathbb{R}$ satisfying

$$(2.2) \quad \|G\|_{\mathcal{L}^2(d\varphi)}^2 := \sum_{j,k=1}^2 \int_{\mathbb{R}} g_j \bar{g}_k d\varphi_{jk} < \infty.$$

In view of Theorem 2, the map $(q_n)_{n \in \mathbb{Z}} \mapsto \begin{pmatrix} \sum_{n \in \mathbb{Z}} q_n u_n(E) \\ \sum_{n \in \mathbb{Z}} q_n v_n(E) \end{pmatrix}$ defines a unitary transformation between $\ell^2(\mathbb{Z})$ and $\mathcal{L}^2(d\mu)$. We call it as the **classical spectral transformation**.

By Chapter V of [26] (Page 297), we know that the matrix of measures $(d\mu_{jk})_{j,k=1,2}$ is Hermitian-positive, and therefore each $d\mu_{jk}$ is a.c. with respect to the measure $d\mu_{11} + d\mu_{22}$. This measure is a.c. with respect to the above spectral measure $\mu_\theta = \mu_{\theta, e-1} + \mu_{\theta, e0}$ and it determines the spectral type of the operator. In particular, if the spectrum of L is purely a.c., we have, for any $q \in \ell^2(\mathbb{Z}) \setminus \{0\}$, the classical spectral transformation is supported on a subset of $\sigma(L)$ with positive Lebesgue measure.

For the classical spectral transformation, there are some singularities with respect to E . More precisely, u_n and v_n are not well differentiated somewhere in the spectrum $\sigma(L)$. For example, for the free Schrödinger operator $(Lq)_n = -(q_{n+1} + q_{n-1})$, we have $\sigma(L) = [-2, 2]$ and for $E \in \sigma(L)$ the rotation number is

$$\xi_0(E) = \rho_{(\omega, A_{(E,0)})}(E) = \cos^{-1} \left(-\frac{E}{2} \right) \in [0, \pi].$$

Since $-E = 2 \cos \xi_0$, we can see that the two generalized eigenvectors

$$(2.3) \quad u_n = \frac{\sin n \xi_0}{\sin \xi_0}, \quad v_n = -\frac{\sin(n-1) \xi_0}{\sin \xi_0}$$

satisfy $\begin{pmatrix} u_1 & v_1 \\ u_0 & v_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and, on $(-2, 2)$, $\xi'_0 = \frac{1}{2 \sin \xi_0}$. Differentiating u_n , we have

$$u'_n = \frac{1}{2 \sin \xi_0} \left(\frac{n \cos n \xi_0}{\sin \xi_0} - \frac{\sin n \xi_0 \cdot \cos \xi_0}{\sin^2 \xi_0} \right).$$

The singularity comes when ξ_0 approaches 0 and π .

2.2. Denominators of continued fraction expansion. Define as usual for $0 < \alpha < 1$,

$$a_0 = 0, \quad \alpha_0 = \alpha,$$

and inductively for $k \geq 1$, as the fractional part of

$$a_k := \max\{n \in \mathbb{Z} : n \leq \alpha_{k-1}^{-1}\}, \quad \alpha_k := \alpha_{k-1}^{-1} - a_k.$$

Then we define

$$p_0 = 0, \quad q_0 = 1, \quad p_1 = a_1, \quad q_1 = 1,$$

and inductively,
$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{cases}.$$

Recall that the sequence (q_n) is the sequence of best denominators of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ since it satisfies

$$\|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}}, \quad \forall 1 \leq k \leq q_n,$$

and $\|q_n \alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}$, where $\|x\|_{\mathbb{T}} := \inf_{p \in \mathbb{Z}} |x - p|$ for $x \in \mathbb{R}$.

2.3. Regularity in the sense of Whitney. Given a closed subset S of \mathbb{R} and $r \in \mathbb{Z}_+$. We give a precise definition of C^r in the sense of Whitney, corresponding to a more general definition in [27].

Definition 1 (C^r in the sense of Whitney). *Given $r+1$ functions $F_k : S \rightarrow \mathbb{C}$ (or $\mathbb{M}(2, \mathbb{C})$), $k = 0, \dots, r$, and some $0 < M < \infty$, such that for $k = 0, \dots, r$,*

$$(2.4) \quad |F_k(x)| \leq M, \quad |F_k(x) - P_k(x, y)| \leq M|x - y|^{1 - \frac{k}{r+1}}, \quad \forall x, y \in S,$$

where $P_k(x, y) := \sum_{k+l \leq r} \frac{1}{l!} F_{k+l}(y)(x - y)^l$. We say that F_0 is C^r in the sense of Whitney on S , denoted by $F_0 \in C_W^r(S)$, with the k^{th} -order derivative F_k , $k = 1, \dots, r$. The $C_W^r(S)$ -norm of F_0 is defined as

$$|F_0|_{C_W^r(S)} := \inf M.$$

REMARK 3. By Whitney's extension theorem[30], we can find an extension $\tilde{F} : \mathbb{R} \rightarrow \mathbb{C}$, which is C^r on \mathbb{R} in the natural sense, such that $\tilde{F}|_S = F_0$ and $\tilde{F}^{(k)}|_S = F_k$. Indeed, the estimation for the upper bound of the $C_W^r(S)$ -norm can be realized by estimating the extension.

- 2.4. Notations.** 1) For any $x \in \mathbb{R}$, let $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ and $\|x\| := \|x\|_{\mathbb{T}} = \inf_{p \in \mathbb{Z}} |x - p|$.
 2) For any subset $S \subset \mathbb{R}$, let $|S|$ be its Lebesgue measure, $cl\{S\}$ be its closure and for any $r > 0$, $B(S, r) := \{x \in \mathbb{R} : |x - y| < r, \exists y \in S\}$. Given any function (possibly matrix-valued) $f(E)$ on \mathbb{R} , let $\text{supp}(f)$ be its support, and ∂_E^k be its k^{th} -order derivative (possibly in the sense of Whitney).
 3) For $h > 0$, let $\mathbb{T}_h := \{z \in \mathbb{C}/\mathbb{Z} : |\Im z| < h\}$. Given any analytic function f on \mathbb{T}_h , let $\hat{f}(n)$ be its n^{th} -Fourier coefficient, $n \in \mathbb{Z}$.
 4) Given $\alpha \in \mathbb{R}$, for any function $\phi : \mathbb{T}_h \rightarrow \mathbb{C}$ and any $n \in \mathbb{N}$, we define the Birkhoff sum of ϕ over $\theta \mapsto \theta + \alpha$ by

$$\phi^{[n]}(x) := \sum_{k=0}^{n-1} \phi(\theta + k\alpha), \quad \phi^{[-n]}(x) := \sum_{k=-n}^{-1} \phi(\theta + k\alpha).$$

Moreover, we denote $R_\phi := \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$.

- 5) Given a compact set $S \subset \mathbb{R}$, for any function F on $S \times \mathbb{T}_h$, denote

$$|F|_{S,h} := \sup_{E \in S} \sup_{\theta \in \mathbb{T}_h} |F(E, \theta)|.$$

When S is a compact set or a finite union of intervals, denote

$$|F|_{C^j(S),h} := \sup_{E \in S} \sup_{\theta \in \mathbb{T}_h} \left(\sum_{l=0}^j |\partial_E^l F(E, \theta)| \right), \quad j \in \mathbb{Z}_+.$$

For the function F defined on $S \times \mathbb{T}$, we define the norms $|F|_S$ and $|F|_{C^j(S)}$ in a similar fashion, without showing the subscript “ \mathbb{T} ” explicitly. If there is no confusion, we denote $|\cdot|_S$ by $|\cdot|$.

In this paper, in the formulations and proofs of various assertions, we shall encounter constants which maybe depend on various quantities, but independent of the index l which represents the iteration step, or the variables (i.e., the phase θ , the energy E and the time t). All such constants will be denoted by c, c_1, c_2, \dots , and sometimes even different constants will be denoted by the same symbol if there is no ambiguity. Moreover, in some estimates, we use the notation “ \lesssim ” or “ \gtrsim ” instead of showing the numerical constant explicitly.

3. ADMISSIBLE SUBSEQUENCES OF DENOMINATORS

In this section, we will explain how to choose subsequences of denominators of an irrational number, which will later serve as indices of resonance in the KAM scheme. As an application, we construct two subsequences depending on a given parameter $\eta > 0$, so that for any T sufficiently large, the interval of the form $[T^\eta, T]$ will be well contained in one interval defined by these subsequences (see the definitions of \mathcal{M}_l and $\tilde{\mathcal{M}}_l$ in Subsection 5.5 for detail).

Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, with (q_n) its sequence of best denominators.

Definition 2 ($CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ bridge, Definition 3.1 in [6]). *Given $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$. For integers $1 \leq l < n$, we say that the pair of denominators (q_l, q_n) forms a $CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ bridge if*

- $q_{i+1} \leq q_i^{\mathcal{A}}, \forall i = l, \dots, n-1;$
- $q_l^{\mathcal{C}} \geq q_n \geq q_l^{\mathcal{B}}.$

Given $0 < \mathcal{A} \leq \mathcal{B}' \leq \mathcal{B} \leq \mathcal{C} \leq \mathcal{C}'$, it is obvious that any $CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ bridge is also a $CD(\mathcal{A}, \mathcal{B}', \mathcal{C}')$ bridge.

For any subsequence (q_{n_k}) of (q_n) , we follow the notations in [6] and denote subsequences $(Q_k) = (q_{n_k})$ and $(\bar{Q}_k) = (q_{n_k+1})$. The properties of these notations have been exploited in Proposition 3.1 in [6]. We will recall it later.

Definition 3 ($(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ –admissible). Given $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$ and $\mathcal{D} > 0$, a subsequence (Q_k) is called $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ –admissible if for any $k \geq 1$

- $Q_k \leq \overline{Q}_{k-1}^{\mathcal{D}}$;
- either $\overline{Q}_k > Q_k^{\mathcal{A}}$ or $(\overline{Q}_{k-1}, Q_k), (Q_k, Q_{k+1})$ are both $CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ bridges.

Given $0 < \mathcal{A} \leq \mathcal{B}' \leq \mathcal{B} \leq \mathcal{C} \leq \mathcal{C}'$ and $0 < \mathcal{D} \leq \mathcal{D}'$, it is obvious that any $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ –admissible subsequence is also $(\mathcal{A}, \mathcal{B}', \mathcal{C}', \mathcal{D}')$ –admissible.

LEMMA 2 (essentially Lemma 3.2 of [6]). Given any $\mathcal{A} > 1$, there exists an $(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21}, \mathcal{A}^{20})$ –admissible subsequence (Q_k) with $Q_0 = 1$.

REMARK 4. In Lemma 3.2 of [6], the authors showed the existence of $(\mathcal{A}, \mathcal{A}, \mathcal{A}^3, \mathcal{A}^4)$ –admissible subsequences. Lemma 2 can be shown following their proof with obvious modifications.

In particular, from the definition of $(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21}, \mathcal{A}^{20})$ –admissible subsequence, we can easily see that $Q_{k+1} \geq Q_k^{\mathcal{A}}$ for each k . Indeed,

- If $\overline{Q}_k > Q_k^{\mathcal{A}}$, it is obvious since $\overline{Q}_k \leq Q_{k+1}$.
- If $\overline{Q}_k \leq Q_k^{\mathcal{A}}$, we have (Q_k, Q_{k+1}) is a $CD(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21})$ bridge, so $Q_{k+1} \geq Q_k^{\mathcal{A}^3} \geq Q_k^{\mathcal{A}}$.

LEMMA 3. Let $\mathcal{A} > 1$ and (Q_k) be an $(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21}, \mathcal{A}^{20})$ –admissible subsequence. There exists a subsequence (R_k) such that $R_0 = 1$, and

- (1) $Q_k^{\mathcal{A}} \leq R_k^{\mathcal{A}} \leq Q_{k+1}$ and $\overline{R}_k \geq Q_k^{\mathcal{A}}$ for each $k \geq 1$;
- (2) (R_k) is $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ –admissible.

Proof. For $k \geq 1$, define R_k as the largest denominator such that $Q_k \leq R_k < Q_k^{\mathcal{A}}$. Then we have $\overline{R}_k \geq Q_k^{\mathcal{A}}$ since $\overline{R}_k > R_k$. We have two cases :

- If $\overline{Q}_k > Q_k^{\mathcal{A}}$, we can see $R_k = Q_k$. So $R_k^{\mathcal{A}} \leq \overline{Q}_k (= \overline{R}_k) \leq Q_{k+1}$.
- If $\overline{Q}_k \leq Q_k^{\mathcal{A}}$, we have that $Q_{k+1} \geq Q_k^{\mathcal{A}^3} \geq R_k^{\mathcal{A}^2}$.

In both case we have $Q_{k+1} \geq R_k^{\mathcal{A}}$. (1) is proven.

It remains to verify that (R_k) is $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ –admissible. By the property of (Q_k) and the definition of (R_k) , we have

$$(3.1) \quad R_k \leq Q_k^{\mathcal{A}} \leq \overline{Q}_{k-1}^{\mathcal{A}^{21}} \leq \overline{R}_{k-1}^{\mathcal{A}^{21}} \text{ for every } k \geq 1.$$

Assume that $\overline{R}_k \leq R_k^{\mathcal{A}}$ for some k (otherwise we can finish the proof without showing that $(\overline{R}_{k-1}, R_k), (R_k, R_{k+1})$ are both $CD(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22})$ bridges). As shown above, $\overline{Q}_k > Q_k^{\mathcal{A}}$ implies $\overline{R}_k > R_k^{\mathcal{A}}$, so we have $\overline{Q}_k \leq Q_k^{\mathcal{A}}$. Thus $(\overline{Q}_{k-1}, Q_k), (Q_k, Q_{k+1})$ are both $CD(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21})$ bridges.

By (1), we have $\overline{Q}_{k-1} \leq \overline{R}_{k-1} \leq Q_k \leq R_k$. Since $(\overline{Q}_{k-1}, Q_k)$ is a $CD(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21})$ bridge, we have that $R_k \geq Q_k \geq \overline{Q}_{k-1}^{\mathcal{A}^3}$ and

$$(3.2) \quad q_{i+1} \leq q_i^{\mathcal{A}} \text{ for every } \overline{Q}_{k-1} \leq q_i < Q_k.$$

Moreover, since $R_k < Q_k^{\mathcal{A}}$, we know that

$$(3.3) \quad q_{i+1} \leq q_i^{\mathcal{A}} \text{ for every } Q_k \leq q_i < R_k.$$

If $\overline{Q}_{k-1} > Q_{k-1}^{\mathcal{A}}$, then $R_{k-1} = Q_{k-1}$ and as a result $\overline{R}_{k-1} = \overline{Q}_{k-1}$. Otherwise we have that $\overline{R}_{k-1} \leq R_{k-1}^{\mathcal{A}} < Q_{k-1}^{\mathcal{A}^2}$. In both case we have $\overline{R}_{k-1} \leq \overline{Q}_{k-1}^{\mathcal{A}^2}$. Thus $\overline{R}_{k-1}^{\mathcal{A}} \leq \overline{Q}_{k-1}^{\mathcal{A}^3} \leq R_k$. Combing with (3.1), (3.2) and (3.3), we get that $(\overline{R}_{k-1}, R_k)$ is a $CD(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22})$ bridge.

By the property of the $CD(\mathcal{A}, \mathcal{A}^3, \mathcal{A}^{21})$ bridge (Q_k, Q_{k+1}) , together with the definition of (R_k) , we have $R_k^{A^2} \leq Q_k^{A^3} \leq Q_{k+1} \leq R_{k+1} \leq Q_{k+1}^A \leq Q_k^{A^{22}} \leq R_k^{A^{22}}$ and

$$q_{i+1} \leq q_i^A \quad \text{for every } R_k \leq q_i < Q_{k+1}.$$

Since $R_{k+1} \leq Q_{k+1}^A$, we also have

$$q_{i+1} \leq q_i^A \quad \text{for every } Q_{k+1} \leq q_i < R_{k+1}.$$

Thus (R_k, R_{k+1}) is a $CD(\mathcal{A}, \mathcal{A}, A^{22})$ bridge. This completes the proof of (2). \square

The main property of admissible sequences we will use is summarised in the following proposition, which is essentially contained in Proposition 3.1 in [6].

PROPOSITION 1 (Proposition 3.1 in [6]). *Given any $\eta \in (0, 1)$, $h_* > 0$ and $M > 1$, there exists $\mathcal{A}_1 = \mathcal{A}_1(M) > 0$ such that for any $\mathcal{A} > \mathcal{A}_1$, there exists $C(h_*, \eta, \mathcal{A}) > 0$, such that for any irrational α , any $(\mathcal{A}, \mathcal{A}, A^{22}, A^{21})$ -admissible subsequence (Q_k) of denominators of α with $Q_0 = 1$, and any $h > h_*$, any function $\phi \in C_h^\omega(\mathbb{T}, \mathbb{R})$, it holds for any $k \in \mathbb{Z}_+$ and $h_k := h(1 - \eta k^{-2})$,*

- $\|\phi^{[Q_k]} - Q_k \hat{\phi}(0)\|_{h_k} \leq C \|\phi - \hat{\phi}(0)\|_h (Q_k^{-M} + \overline{Q}_k^{-1+\frac{1}{M}}),$
- for any $0 \leq l \leq Q_{k+1}$, $\|\phi^{[l]} - l \hat{\phi}(0)\|_{h_k} \leq C \|\phi - \hat{\phi}(0)\|_h (\overline{Q}_k Q_k^{-M} + \overline{Q}_k^{\frac{1}{M}}).$

4. A KAM SCHEME FOR $SL(2, \mathbb{R})$ COCYCLES

In this section, we recall the KAM scheme for $SL(2, \mathbb{R})$ cocycles developed in [6].

Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $(q_n)_{n \geq 0}$ the sequence of denominators, and an open interval $J \subset \mathbb{R}$. For $A : J \times \mathbb{T} \rightarrow SL(2, \mathbb{R})$, analytic on $J \times \mathbb{T}_h$ for some $h > 0$, we consider the cocycle (α, A) :

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A(E, \theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

For $n \geq 1$, we denote the iterates of (α, A) by

$$\begin{aligned} A^{(n)}(E, \theta) &:= A(E, \theta + (n-1)\alpha) \cdots A(E, \theta), \\ A^{(-n)}(E, \theta) &:= A(E, \theta - n\alpha)^{-1} \cdots A(E, \theta - \alpha)^{-1}. \end{aligned}$$

Denote $\rho(E) := \rho_{(\alpha, A(E, \cdot))}$ the fibered rotation number.

Before stating the main result in this section, we introduce some necessary objects. Given $\tau > 0$, $0 < \nu < \frac{1}{2}$ and $\varepsilon > 0$, we denote

$$\mathcal{Q}_\alpha(\varepsilon, \tau, \nu) := \{\rho \in \mathbb{T} : \|2q_n \rho\| > \varepsilon \max(q_n^{-\tau}, q_{n+1}^{-\nu}), \forall n \in \mathbb{N}\}.$$

We call $\mathcal{Q}_\alpha(\varepsilon, \tau, \nu)$ the set of **non-resonant fibered rotation numbers** with parameters ε, τ, ν . For any subsequence (Q_k) of denominators, define

$$\mathcal{Q}_l := \{\rho \in \mathbb{T} : \|2Q_l \rho\| > \varepsilon \max(Q_l^{-\tau}, \overline{Q}_l^{-\nu})\}, \quad \overline{\mathcal{Q}}_l := \cap_{k=1}^l \mathcal{Q}_k, \quad \overline{\mathcal{Q}} := \cap_{k=1}^\infty \mathcal{Q}_k,$$

and $\Omega_l := \rho^{-1}(\mathcal{Q}_l)$, $\overline{\Omega}_l := \rho^{-1}(\overline{\mathcal{Q}}_l)$, $\overline{\Omega} := \rho^{-1}(\overline{\mathcal{Q}}) = \cap_{k=1}^\infty \Omega_k$.

PROPOSITION 2. *Given $h > 0$, an open interval $J \subset \mathbb{R}$, let $A : J \times \mathbb{T} \rightarrow SL(2, \mathbb{R})$, analytic on $J \times \mathbb{T}_h$, and $A_0 \in SO(2, \mathbb{R})$. Given $\tau > 0$, $0 < \nu < \frac{1}{2}$, $\varepsilon > 0$, integer $r \geq 1$, there exists $\mathcal{A}_0 = \mathcal{A}_0(\tau, \nu) > 0$, such that for any $\mathcal{A} > \mathcal{A}_0$, one can find $\epsilon_0 = \epsilon_0(\tau, \nu, \varepsilon, h, r, \mathcal{A}) > 0$ such that if $|A - A_0|_{J, h} < \epsilon_0$, then the following is true for any $(\mathcal{A}, \mathcal{A}, A^{22}, A^{21})$ -admissible sequence (Q_k) .*

(1) There exist $\begin{cases} W : (J \cap \overline{\Omega}) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \phi : (J \cap \overline{\Omega}) \times \mathbb{T} \rightarrow \mathbb{T} \end{cases}$, such that for each $E \in J \cap \overline{\Omega}$, $W(E, \cdot)$ and $\phi(E, \cdot)$ are analytic, satisfying

$$(4.1) \quad A(E, \theta) = W(E, \theta + \alpha) R_{\phi(E, \theta)} W(E, \theta)^{-1}.$$

(2) For $l \geq 1$, there exist $\begin{cases} W_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \phi_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow \mathbb{T} \\ \xi_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow \mathbb{M}(2, \mathbb{R}) \end{cases}$, analytic on $(J \cap \overline{\Omega}_l) \times \mathbb{T}$, such that

$$A(E, \theta) = W_l(E, \theta + \alpha) R_{\phi_l(E, \theta)} (id + \xi_l(E, \theta)) W_l(E, \theta)^{-1}$$

and there exist constants $c_1, D_1 > 0$, such that for every $l \geq 1$,

$$(4.2) \quad |W|_{J \cap \overline{\Omega}}, |\phi|_{J \cap \overline{\Omega}}, |W_l|_{C^r(J \cap \overline{\Omega}_l)}, |\phi_l|_{C^r(J \cap \overline{\Omega}_l)} < c_1,$$

$$(4.3) \quad |W_l - W_{l+1}|_{C^r(J \cap \overline{\Omega}_{l+1})}, |\phi_l - \phi_{l+1}|_{C^r(J \cap \overline{\Omega}_{l+1})} < c_1 e^{-\overline{Q}_l^{D_1}},$$

$$(4.4) \quad |\xi_l|_{C^r(J \cap \overline{\Omega}_l)} < c_1 e^{-\overline{Q}_l^{D_1}}.$$

(3) If there exists $m_1 \in \mathbb{Z}_+$ and $\Lambda \in C^\omega(\mathbb{T}, SL(2, \mathbb{R}))$ such that

$$(4.5) \quad \inf_{\substack{v \in \mathbb{P}(\mathbb{R}^2) \\ E \in J, \theta \in \mathbb{T}}} \partial_E (\Lambda(\theta + m_1 \alpha) A^{(m_1)}(E, \theta) \Lambda(\theta)^{-1} v) > 0,$$

then there exists $C_1 > 0$ and $n_0, l_0 \in \mathbb{Z}_+$ such that for any $n > n_0, l > l_0$,

$$\partial_E \phi_l^{[n]}(E, \theta) > C_1 n, \quad \forall E \in J \cap \overline{\Omega}_l, \theta \in \mathbb{T}.$$

Proof of Proposition 2: To prove Proposition 2, the main step is to construct W_l and ϕ_l by induction. By letting l tend to infinity, we shall obtain W and ϕ as their limits.

Given any constants $\kappa \in (0, 1)$, $D > 1$, we define the sequences

$$\kappa_l := \frac{\kappa}{l^2}, \quad D_l := D - \kappa_l, \quad h_l := h \prod_{i=1}^l (1 - \kappa_i)^2.$$

Let $h_\infty := \lim_{l \rightarrow \infty} h_l$. It is clear that $h_\infty > 0$. Let $U_k := e^{-\overline{Q}_k Q_k^{-b} - \overline{Q}_k^a}$, where $a = \frac{2}{M}$, $b = a^{-1}$, and

$$M = \max \left(4(\tau + 1), \frac{4}{1 - 2\nu} \right), \quad \mathcal{A}_0 = \mathcal{A}_1(M),$$

with $\mathcal{A}_1(M)$ defined as in Proposition 1. The inductive step is given by the following lemma.

LEMMA 4 (Inductive lemma). *Given any $\kappa \in (0, 1)$, $D > 1$, any $r \in \mathbb{Z}_+$, there exists $T_0(h, D, \kappa, \varepsilon, \nu, \tau, r) > 0$, $C_0(h, D, \kappa, \varepsilon, \nu, \tau, r) > 0$, such that the following is true.*

For any l such that $Q_l \geq T_0$, if there exist $\begin{cases} W_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \phi_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow \mathbb{T} \\ \xi_l : (J \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow \mathbb{M}(2, \mathbb{R}) \end{cases}$, analytic on $(J \cap \overline{\Omega}_l) \times \mathbb{T}_{h_l}$, satisfying $|\phi_l - \hat{\phi}_l(0)|_{J \cap \overline{\Omega}_l, h_l} < D_l$, $|\xi_l|_{J \cap \overline{\Omega}_l, h_l} < U_l$, and

$$A(E, \theta) = W_l(E, \theta + \alpha) R_{\phi_l(E, \theta)} (id + \xi_l(E, \theta)) W_l(E, \theta)^{-1},$$

then there exist $\begin{cases} B_l : (J \cap \overline{\Omega}_{l+1}) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \phi_{l+1} : (J \cap \overline{\Omega}_{l+1}) \times \mathbb{T} \rightarrow \mathbb{R} \\ \xi_{l+1} : (J \cap \overline{\Omega}_{l+1}) \times \mathbb{T} \rightarrow \mathbb{M}(2, \mathbb{R}) \end{cases}$, analytic on $(J \cap \overline{\Omega}_{l+1}) \times \mathbb{T}_{h_{l+1}}$, such that $W_{l+1} = W_l B_l^{-1}$ satisfies

$$A(E, \theta) = W_{l+1}(E, \theta + \alpha) R_{\phi_{l+1}(E, \theta)} (id + \xi_{l+1}(E, \theta)) W_{l+1}(E, \theta)^{-1},$$

with $|\phi_{l+1} - \hat{\phi}_{l+1}(0)|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}} < D_{l+1}$ and $|\xi_{l+1}|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}} < U_{l+1}$. Moreover if

$$(1 - 2^{-l})H_0 > 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \phi_l|_{J \cap \overline{\Omega}_l, h_l}^{\frac{r}{j}}, |\partial_E^j \xi_l|_{J \cap \overline{\Omega}_l, h_l}^{\frac{r}{j}} U_l^{-\frac{r}{j}} \right),$$

for some constant H_0 , then for $0 \leq j \leq r$,

$$|\partial_E^j (B_l - id)|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}}, |\partial_E^j (\phi_{l+1} - \phi_l)|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}} \leq C_0 U_l^{\frac{1}{3}} H_0^{\frac{l}{r}},$$

and $(1 - 2^{-l-1})H_0 > 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \phi_{l+1}|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}}^{\frac{r}{j}}, |\partial_E^j \xi_{l+1}|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}}^{\frac{r}{j}} U_{l+1}^{-\frac{r}{j}} \right)$.

Proof of Lemma 4. Let r given as Lemma 4. Lemma 4 is a consequence of the following three lemmas.

LEMMA 5. [Lemma 4.5 of [6]] Let ϕ_l, ξ_l be given as in Lemma 4 and denote $A_l = R_{\phi_l}(id + \xi_l)$. There exists $T = T(h, D, \kappa, \varepsilon, \nu, \tau, r)$ such that if $Q_l \geq T$, then, on $\overline{\Omega}_{l+1}$, $A_l^{(Q_{l+1})}$ can be expressed as $R_{\phi_l^{[Q_{l+1}]}}(id + \xi_{(Q_{l+1})})$ with

$$\begin{aligned} \left| \phi_l^{[Q_{l+1}]} - \widehat{\phi_l^{[Q_{l+1}]}}(0) \right|_{J \cap \overline{\Omega}_{l+1}, h_l(1-\kappa_{l+1})} &\leq D, \\ |(R_{2\phi_l^{[Q_{l+1}]}} - id)^{-1}|_{J \cap \overline{\Omega}_{l+1}, h_l(1-\kappa_{l+1})} &< \varrho_l^{-1} := \frac{\varepsilon}{4(Q_{l+1}^{-\tau} + \overline{Q}_{l+1}^{-\nu})} < U_l^{-\frac{1}{40r}}, \\ |\xi_{(Q_{l+1})}|_{J \cap \overline{\Omega}_{l+1}, h_l(1-\kappa_{l+1})} &\leq U_l^{\frac{1}{2}}. \end{aligned}$$

Moreover, if for $\overline{H} > 0$ we have $\overline{H} > 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \phi_l|_{J \cap \overline{\Omega}_l, h_l}^{\frac{r}{j}}, |\partial_E^j \xi_l|_{J \cap \overline{\Omega}_l, h_l}^{\frac{r}{j}} U_l^{-\frac{r}{j}} \right)$, then

$$\overline{H} Q_{l+1}^r \gtrsim 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \phi_l^{[Q_{l+1}]}|_{J \cap \overline{\Omega}_{l+1}, h_l(1-\kappa_{l+1})}^{\frac{r}{j}}, |\partial_E^j \xi_{(Q_{l+1})}|_{J \cap \overline{\Omega}_{l+1}, h_l(1-\kappa_{l+1})}^{\frac{r}{j}} U_l^{-\frac{r}{2j}} \right).$$

Proof. The first three inequalities is contained in Lemma 4.5 in [6]. The last inequality follows from direct computation. Here we have used the definition of M and Proposition 1. \square

LEMMA 6. For every $D, h_* > 0$, there exist $\epsilon = \epsilon(D, h_*) > 0$, $C_0 = C_0(D, h_*) > 0$ such that the following is true.

Given any $\epsilon_0 \in (0, \epsilon)$, $h > h_*$, $0 < \delta < 1$, $\bar{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$, given some open interval $J_0 \subset \mathbb{R}$, and $\begin{cases} \bar{A} : J_0 \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \bar{\varphi} : J_0 \times \mathbb{T} \rightarrow \mathbb{R} \end{cases}$, analytic on $J_0 \times \mathbb{T}_h$, with

$$|\bar{\varphi} - \widehat{\bar{\varphi}}(0)|_{J_0, h} \leq D, \quad \varrho^{-1} := \max(4, |(R_{2\bar{\varphi}} - id)^{-1}|_{J_0, h}) < \epsilon_0^{-\frac{1}{20r}}, \quad |R_{-\bar{\varphi}} \bar{A} - id|_{J_0, h} < \epsilon_0.$$

Denote $\tilde{\xi} = R_{-\bar{\varphi}} \bar{A} - id$. Then there exist $\begin{cases} B : J_0 \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \varphi : J_0 \times \mathbb{T} \rightarrow \mathbb{T} \end{cases}$, analytic on $J_0 \times \mathbb{T}_{e^{-\delta/5h}}$, such that for

$$\tilde{A}(E, \theta) := B(E, \theta + \bar{\alpha}) \bar{A}(E, \theta) B(E, \theta)^{-1}$$

and any constant $H \geq 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \bar{\varphi}|_{J_0, h}^{\frac{r}{j}}, |\partial_E^j \bar{\xi}|_{J_0, h}^{\frac{r}{j}} \epsilon_0^{-\frac{r}{j}} \right)$, we have

$$(4.6) \quad |\partial_E^j (B - id)|_{J_0, e^{-\delta/5} h} < C_0 \epsilon_0 \rho^{-j-1} H^{\frac{j}{r}}, \quad j = 0, \dots, r,$$

$$(4.7) \quad |R_{-\varphi} \tilde{A} - id|_{J_0, e^{-\delta/5} h} < C_0 \epsilon_0 e^{-\frac{\delta h \epsilon_0^{2r+1}}{C_0 \|\bar{\alpha}\|}},$$

$$(4.8) \quad |\mathcal{Q}(\partial_E^j \tilde{A})|_{J_0, e^{-\delta/5} h} < C_0 \epsilon_0 e^{-\frac{\delta h \epsilon_0^{2r+1}}{C_0 \|\bar{\alpha}\|}} H^{\frac{j}{r}}, \quad j = 1, \dots, r.$$

where $\mathcal{Q}(P) := \frac{P+JPJ}{2}$ for any $P \in \mathbb{M}(2, \mathbb{R})$, with $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We leave the proof of Lemma 6 to the appendix.

By Lemma 5, we can apply Lemma 6 to $\bar{A} = A_l^{(Q_{l+1})}$, $\bar{\alpha} = Q_{l+1} \alpha$ and $\bar{\varphi} = \phi_l^{[Q_{l+1}]}$, with J_0 a connected component of $J \cap \bar{\Omega}_{l+1}$, $\epsilon_0 = U_l^{\frac{1}{2}}$, $h = h_l(1 - \kappa_{l+1})$, $h_* = h_\infty$, $H = c_0 H_0 Q_{l+1}^r$ for some large absolute constant $c_0 > 0$, and $\delta = -\log(1 - \kappa_{l+1})$, we obtain $B_l = B$. The upper bound of $|\partial_E^j (B_l - id)|_{J \cap \bar{\Omega}_{l+1}, h_{l+1}}$ follows from (4.6), $\rho^{20r} \geq \epsilon_0$ and $T_0 \leq Q_{l+1} \leq \bar{Q}_l^{A^{21}}$.

LEMMA 7. Let $h > h_*$, $D, \delta, \epsilon_0, \varphi, H, \bar{\alpha}, \bar{A}, B, \varphi, \tilde{A}$ be given as in Lemma 6. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists $D_0(h_*, D) > 0$ such that the following is true.

For any $G : J_0 \times \mathbb{T} \rightarrow SL(2, \mathbb{R})$, which is analytic on $J_0 \times \mathbb{T}_h$ and satisfying $|G|_{J_0, h} \leq D$, $G = R_{\tilde{\zeta}}(id + \tilde{\zeta})$ for some $\begin{cases} \tilde{\zeta} : J_0 \times \mathbb{T} \rightarrow \mathbb{T} \\ \tilde{\zeta} : J_0 \times \mathbb{T} \rightarrow \mathbb{M}(2, \mathbb{R}) \end{cases}$, analytic on $J_0 \times \mathbb{T}_h$, with $|\tilde{\zeta}|_{J_0, h} < \epsilon_0$, and

$$H > 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \tilde{\zeta}|_{J_0, h}^{\frac{r}{j}}, |\partial_E^j \tilde{\xi}|_{J_0, h}^{\frac{r}{j}} \epsilon_0^{-\frac{r}{j}} \right).$$

Moreover we have $G(\theta + \bar{\alpha}) \bar{A}(\theta) = \bar{A}(\theta + \alpha) G(\theta)$. Then there exist $\begin{cases} \tilde{\zeta} : J_0 \times \mathbb{T} \rightarrow \mathbb{T} \\ \tilde{\zeta} : J_0 \times \mathbb{T} \rightarrow \mathbb{M}(2, \mathbb{R}) \end{cases}$, analytic on $J_0 \times \mathbb{T}_{e^{-\delta} h}$, such that $\tilde{G}(\theta) := B(\theta + \alpha) G(\theta) B(\theta)^{-1}$ can be expressed as $\tilde{G} = R_{\tilde{\zeta}}(id + \tilde{\zeta})$. Moreover, for $j = 0, \dots, r$, we have

$$(4.9) \quad |\partial_E^j (\tilde{\zeta} - \zeta)|_{J_0, e^{-\delta} h} < D_0 H^{\frac{j}{r}} \epsilon_0, \quad |\partial_E^j \tilde{\xi}|_{J_0, e^{-\delta} h} < D_0 H^{\frac{j}{r}} e^{-\frac{\delta h \epsilon_0^{2r+1}}{D_0 \|\bar{\alpha}\|}}.$$

We leave the proof of Lemma 7 to the Appendix A.

Recall that $\epsilon_0 = U_l^{\frac{1}{2}}$. Apply Lemma 7 to $G = A_l$, $\zeta = \phi_l$, $\tilde{\zeta} = \xi_l$, with J_0 a connected component of $J \cap \bar{\Omega}_{l+1}$, we obtain $\phi_{l+1} = \tilde{\zeta}$, $\xi_{l+1} = \tilde{\xi}$ for Lemma 4 and establish the desired estimates for $|\partial_E^j (\phi_{l+1} - \phi_l)|$, $j = 0, \dots, r$, by the first inequality in (4.9) and $U_l^{1/6} \ll Q_{l+1}^{-r}$. Moreover, the second inequality in (4.9) yields that

$$|\xi_{l+1}|_{J \cap \bar{\Omega}_{l+1}, h_{l+1}} \leq D_0 e^{-\frac{h(1-\kappa_{l+1})\kappa_{l+1}e_l^{2r+1}}{D_0 \|Q_{l+1}\alpha\|}} < U_{l+1},$$

in view of the proof of Proposition 4.7 in [6].

Finally, we verify the last statement in Lemma 4. Notice that by the second inequality in (4.9), we have, again by the proof of Proposition 4.7 in [6], for all $1 \leq j \leq r$, $Q_l \geq T_0$ for sufficiently large T_0 ,

$$|\partial_E^j \xi_{l+1}|_{J_0 \cap \bar{\Omega}_{l+1}, h_{l+1}} < D_0 H_0^{\frac{j}{r}} (c_0 Q_{l+1}^r)^{\frac{j}{r}} e^{-\frac{h(1-\kappa_{l+1})\kappa_{l+1}e_l^{2r+1}}{D_0 \|Q_{l+1}\alpha\|}} < D_0 H_0^{\frac{j}{r}} U_{l+1}.$$

□

Now we are ready to prove Proposition 2. Recall that we have given $0 < \kappa < 1$, $D > 1$ at the beginning of proof and $\tau > 0$, $0 < \nu < \frac{1}{2}$, $\varepsilon > 0$, $h > 0$. We choose $T_0 = T_0(h, D, \kappa, \varepsilon, \nu, \tau, r) > 0$ as in Lemma 4. Let $\epsilon_0 < U_{l_0}$, where l_0 is the smallest number such that $Q_{l_0} > T_0$ with (Q_k) the $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible subsequence. We can see that ϵ_0 can be chosen only depending on $h, \kappa, \varepsilon, \tau, \nu, \mathcal{A}, r$, while independent of the specific choice of the subsequence (Q_l) .

Since $|A - A_0|_J < \epsilon_0$ for some $A_0 \in SO(2, \mathbb{R})$, let $W_l = id$, $R_{\phi_l} = A_0$ and $\xi_l = R_{-\phi_l} A - id$. We see that ϕ_l is constant in θ , and $|\xi_l|_J < \epsilon_0$. Then we can apply Lemma 4 iteratively, starting on $J \cap \overline{\Omega}_{l_0}$, and obtain the sequences $\{B_l\}$, $\{\phi_l\}$ and $\{\xi_l\}$. In view of Lemma 4, we have, for all $l \geq l_0$,

$$|\partial_E^j(B_l - id)|_{J \cap \overline{\Omega}_{l+1}, h_{l+1}} < C_0 U_l^{\frac{1}{3}} H_l^{\frac{j}{7}}, \quad j = 0, \dots, r.$$

Thus we can obtain W, ϕ as the limits of W_l, ϕ_l respectively as $l \rightarrow \infty$. By Lemma 4, we obtain (4.3), (4.4), and hence (4.2). Of course we can construct W_l arbitrarily for $l < l_0$ at the cost of enlarging c_1 in (4.2) – (4.4). Then (1) and (2) are proven.

We now prove (3). Define $C_2 > 0$ as

$$C_2 := \inf_{\substack{v \in \mathbb{P}(\mathbb{R}^2) \\ E \in J, \theta \in \mathbb{T}}} \partial_E(\Lambda(\theta + m_1 \alpha) A^{(m_1)}(E, \theta) \Lambda(\theta)^{-1} v),$$

Recall that, for each $l \geq 1$, $A(E, \theta) = W_l(E, \theta + \alpha) A_l(E, \theta) W_l(E, \theta)^{-1}$ with

$$(4.10) \quad A_l(E, \theta) = R_{\phi_l(E, \theta)}(id + \xi_l(E, \theta)).$$

So for any $m, l \geq 1$, $E \in J \cap \overline{\Omega}_l$, $\theta \in \mathbb{T}$, $v \in \mathbb{P}(\mathbb{R}^2)$, we have, with $\theta_j := \theta + jm_1 \alpha$ for $j \in \mathbb{Z}$,

$$(4.11) \quad \begin{aligned} & \partial_E(\Lambda(\theta_m) A^{(mm_1)}(E, \theta) \Lambda(\theta)^{-1} v) \\ &= \partial_E(\Lambda W_l)(E, \theta_m) (A_l^{(mm_1)}(E, \theta) (\Lambda W_l)(E, \theta)^{-1} v) \end{aligned}$$

$$(4.12) \quad + D(\Lambda W_l)(E, \theta_m) \left(A_l^{(mm_1)}(E, \theta) (\Lambda W_l)(E, \theta)^{-1} v, \partial_E A_l^{(mm_1)}(E, \theta) (\Lambda W_l)(E, \theta)^{-1} v \right)$$

$$(4.13) \quad + D(\Lambda W_l)(E, \theta_m) A_l^{(mm_1)}(E, \theta) \left(\Lambda W_l(E, \theta)^{-1} v, \partial_E (\Lambda W_l(E, \theta)^{-1} v) \right).$$

Notice that $|D(\Lambda W_l)|_{J \cap \overline{\Omega}_l} \leq |\Lambda|^2 |W_l|_{J \cap \overline{\Omega}_l}^2 \leq c_1^2 |\Lambda|^2$. By (2), we see that, for any l ,

$$(4.14) \quad |(4.11)| \leq c_1 |\Lambda|^2, \quad |(4.13)| \leq c_1^5 |\Lambda|^4 |A_l^{(mm_1)}|_{J \cap \overline{\Omega}_l}^2.$$

By (2), for $m \geq 1$, there exists $l(m) \geq 1$ such that $|A_l^{(mm_1)}|_{J \cap \overline{\Omega}_l} < 2$ for every $l \geq l(m)$. On the other hand, we have

$$(4.15) \quad |(4.12)| \leq c_1^2 |\Lambda|^2 |\partial_E A_l^{(mm_1)}(E, \theta) ((\Lambda W_l)(E, \theta)^{-1} v)|.$$

By (4.1), for any $E \in J \cap \overline{\Omega}$, and $k \in \mathbb{Z}$, we have

$$|D(\Lambda(\theta_k) A^{(km_1)}(E, \theta) \Lambda(\theta)^{-1})| \geq |\Lambda|^{-4} |W|^{-4} \geq |\Lambda|^{-4} c_1^{-4}.$$

Thus, for any $E \in J \cap \overline{\Omega}$, $\theta \in \mathbb{T}$, $v \in \mathbb{P}(\mathbb{R}^2)$,

$$(4.16) \quad \begin{aligned} |\partial_E(\Lambda(\theta_m) A^{(mm_1)}(E, \theta) \Lambda(\theta)^{-1} v)| &= \sum_{k=1}^m \left| D(\Lambda(\theta_m) A^{((m-k)m_1)}(E, \theta_k) \Lambda(\theta_k)^{-1})(v_{k,1}, v_{k,2}) \right| \\ &\geq |\Lambda|^{-4} c_1^{-4} m C_2, \end{aligned}$$

with $v_{k,1} := \Lambda(\theta_k) A^{(km_1)}(E, \theta) \Lambda(\theta)^{-1} v$ and

$$v_{k,2} := \partial_E(\Lambda(\theta_k) A^{(m_1)}(E, \theta_{k-1}) \Lambda(\theta_{k-1})^{-1})(v_{k-1,1}),$$

since $v_{k,2} \geq C_2$ for all k .

Using (4.10), (4.14), (4.15), (4.16) and (4.4), by taking m sufficiently large, we can ensure that for every $l \geq l(m)$,

$$|\partial_E A_l^{(mm_1)}(E, \theta)((\Lambda W_l)(E, \theta)^{-1}v)| > 1, \quad \forall E \in J \cap \overline{\Omega}, \theta \in \mathbb{T}, v \in \mathbb{P}(\mathbb{R}^2).$$

Then we conclude that $\partial_E A_l^{(mm_1)}(E, \theta)v > c_1^{-2}|\Lambda|^{-2}$ for all $E \in J \cap \overline{\Omega}$, $\theta \in \mathbb{T}$ and $v \in \mathbb{P}(\mathbb{R}^2)$. By (2), we see that $|\xi_l|_{C^1(J \cap \overline{\Omega}_l)} \rightarrow 0$ as $l \rightarrow \infty$. Thus by (4.10), $\partial_E \phi_l^{(mm_1)}(E, \theta) > \frac{1}{2}c_1^{-2}|\Lambda|^{-2}$ for all $E \in J \cap \overline{\Omega}$, $\theta \in \mathbb{T}$ and sufficiently large $l \geq 1$. (3) follows from the uniform C^2 bound of ϕ_l on $J \cap \overline{\Omega}_l$ in (4.2). \square

5. GROWTH OF THE DIFFUSION NORM

Recall the time-dependent Schrödinger equation:

$$(5.1) \quad i \frac{d}{dt} u_n = (L_\theta u)_n = -(u_{n+1} + u_{n-1}) + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}.$$

In this section, we will finish the proof of Theorem 1.

5.1. Ballistic upper bound. For Eq. (5.1), we already have the general ballistic upper bound (Lieb-Robinson bound [25]). The proof can be found in [1] and [12]. For the convenience of readers, we sketch the proof in [12], and give the upper bound of the p^{th} -moment of $u(t)$ for any $p > 0$ under some suitable condition of $u(0)$.

Theorem 3. *Given $p > 0$. For $u(0) \in \mathcal{W}^{p'}(\mathbb{Z})$ with $p' = \begin{cases} p, & \frac{p}{2} = \lfloor \frac{p}{2} \rfloor \\ 2\lfloor \frac{p}{2} \rfloor + 2, & \frac{p}{2} > \lfloor \frac{p}{2} \rfloor \end{cases}$, there is a constant $C_3 > 0$, depending on p and $\langle u(0) \rangle_{p'}$, such that*

$$(5.2) \quad \langle u(t) \rangle_p \leq C_3 t^{\frac{p}{2}}, \quad t \geq 1.$$

Proof. Given $m \in \mathbb{Z}_+$, let $X^{(m)}$ be the multiplication operator from $\mathcal{W}^{2m}(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$: $(X^{(m)}u)_n = n^m u_n$, and for $N \in \mathbb{Z}_+$, we define the bounded operator $X_N^{(m)}$ on $\ell^2(\mathbb{Z})$:

$$(X_N^{(m)}u)_n = \begin{cases} -N^m u_n, & n \leq -N-1 \\ -|n|^m u_n, & -N \leq n \leq 0 \\ n^m u_n, & 0 < n \leq N \\ N^m u_n, & n \geq N+1 \end{cases}.$$

Let $A_N^{(m)} = i[L, X_N^{(m)}] = i(LX_N^{(m)} - X_N^{(m)}L)$. By a straightforward computation, we have

$$(A_N^{(m)}u)_n = \begin{cases} i((n-1)^m - n^m)u_{n-1} + i((n+1)^m - n^m)u_{n+1}, & |n| \leq N-1 \\ i(N^m - (N-1)^m)u_{-N+1}, & n = -N \\ -i((N-1)^m - N^m)u_{N-1}, & n = N \\ 0, & |n| \geq N+1 \end{cases}.$$

Note that $(n+1)^m - n^m = \sum_{j=1}^m C_m^j n^{m-j}$ for any $n \in \mathbb{Z}$ and the constants C_m^j are independent of n . Then we see that $A_N^{(m)}$ converges strongly to the self-adjoint operator from $\mathcal{W}^{2m-2}(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$:

$$(A^{(m)}u)_n = i((n-1)^m - n^m)u_{n-1} + i((n+1)^m - n^m)u_{n+1},$$

as $N \rightarrow \infty$, i.e.,

$$\lim_{N \rightarrow \infty} \|A_N^{(m)}u - A^{(m)}u\|_{\ell^2(\mathbb{Z})} = 0, \quad \forall u \in \mathcal{H}^{2m-2}(\mathbb{Z}).$$

Moreover there is a constant $c' = c'(m)$, such that

$$\|A^{(m)}u\|_{\ell^2(\mathbb{Z})}, \|A_N^{(m)}u\|_{\ell^2(\mathbb{Z})} \leq c' \langle u \rangle_{2m-2}, \quad \forall N \in \mathbb{Z}_+.$$

We consider the Heisenberg time evolution $X_N^{(m)}(t) := e^{itL}X_N^{(m)}e^{-itL}$ on $\ell^2(\mathbb{Z})$ for $N \in \mathbb{Z}_+$. Since L is a bounded self-adjoint operator, we can see that $e^{\pm itL}$ are analytic functions of t . So $X_N^{(m)}(t)$ is analytic in t . Differentiating with respect to t , we have

$$\frac{d}{dt}X_N^{(m)}(t) = e^{itL}(iL)X_N^{(m)}e^{-itL} + e^{itL}X_N^{(m)}(-iL)e^{-itL} = e^{itL}A_N^{(m)}e^{-itL} := A_N^{(m)}(t).$$

For $T > 0$, integrating with respect to t on $[0, T]$, we have

$$X_N^{(m)}(T) - X_N^{(m)} = \int_0^T A_N^{(m)}(t) dt.$$

For any $\psi \in \mathcal{W}^{2m}(\mathbb{Z})$, assume that $e^{-itL}\psi \in \mathcal{W}^{2m-2}$ for any finite t . This is true for $m = 1$ in view of the ℓ^2 -conservation law. Then we can define

$$A^{(m)}(t) := e^{itL}A^{(m)}e^{-itL}.$$

Noting that e^{itL} is unitary on $\ell^2(\mathbb{Z})$, we have $\|A^{(m)}(t)\psi\|_{\ell^2(\mathbb{Z})} \leq c' \langle e^{-itL}\psi \rangle_{2m-2}$ for any $t > 0$. Combining with the strong convergence of $A_N^{(m)}(t)$ to $A^{(m)}(t)$, we get, by dominated convergence theorem, that for any $T > 0$, any $\psi \in \mathcal{H}^{2m}(\mathbb{Z})$,

$$\lim_{N \rightarrow \infty} (X_N^{(m)}(T)\psi - X_N^{(m)}\psi) = \lim_{N \rightarrow \infty} \int_0^T A_N^{(m)}(t)\psi dt = \int_0^T A^{(m)}(t)\psi dt,$$

where the limit is taken in space $\ell^2(\mathbb{Z})$. So we can define for $\psi \in \mathcal{W}^{2m}(\mathbb{Z})$ that $X^{(m)}(T)\psi := \lim_{N \rightarrow \infty} X_N^{(m)}(T)\psi$, with

$$\begin{aligned} \langle e^{-iT L}\psi \rangle_{2m} = \|X^{(m)}(T)\psi\|_{\ell^2(\mathbb{Z})} &\leq \|X^{(m)}\psi\|_{\ell^2(\mathbb{Z})} + \int_0^T \|A^{(m)}(t)\psi\|_{\ell^2(\mathbb{Z})} dt \\ &\leq \langle \psi \rangle_{2m} + c' \int_0^T \langle e^{-itL}\psi \rangle_{2m-2} dt. \end{aligned}$$

Since for $m = 1$, $\langle e^{-itL}\psi \rangle_{2m-2} = \sqrt{2}\|\psi\|_{\ell^2(\mathbb{Z})}$ for any finite t , we can get the linear upper bound for $\langle e^{-itL}\psi \rangle_2$. By the induction, for any $m \in \mathbb{Z}_+$, there is a constant $c_m > 0$, depending on m and $\langle \psi \rangle_j$, $j = 0, \dots, m$, such that

$$\langle e^{-iT L}\psi \rangle_{2m} \leq c_m T^m, \quad \forall 1 \leq T < \infty.$$

This concludes the proof of (5.2) when p is an even integer.

For any $p \in (2m, 2m+2)$, with $u(0) \in \mathcal{W}^{2m+2}$, we have

$$\langle u(t) \rangle_{2m} \leq c_m t^m, \quad \langle u(t) \rangle_{2m+2} \leq c_{m+1} t^{m+1}.$$

By Cauchy inequality $\langle u(t) \rangle_p \leq \langle u(t) \rangle_{2m}^{\frac{2m+2-p}{2}} \langle u(t) \rangle_{2m+2}^{\frac{p-2m}{2}}$, we get (5.2). \square

Theorem 3 gives a stronger upper bound than that in Theorem 1. The rest part of this section is devoted to show that for any $\eta > 0$

$$\lim_{T \rightarrow \infty} \frac{\langle u(t) \rangle_p^2}{t^{p-\eta}} = \infty$$

under the condition of Theorem 1. Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $V \in C^\omega(\mathbb{T}, \mathbb{R})$ such that L_θ has purely a.c. spectrum for a.e. $\theta \in \mathbb{T}$. Let $\mathbb{T}_{ac} \subset \mathbb{T}$ denote this full-measure subset.

5.2. Reducibility for the Schrödinger cocycle. Given any integer $r \geq 3$, we define the constants $\tau, \tau_0, \tilde{\tau}_0, \tau_1, \tau_2 > 100$ such that

$$(5.3) \quad \tilde{\tau}_0 \geq \tau_2 > \frac{10^7 \tau_0}{\eta}, \quad \tau_0 \geq \tau_1, \quad \tau = \min(\tau_0, \tilde{\tau}_0).$$

For any $0 < \nu < \frac{1}{2}$, we choose \mathcal{A} such that

$$(5.4) \quad \mathcal{A} > \max \left(\frac{10^7 \tilde{\tau}_0}{\eta \tau_1}, \frac{10^7 \tilde{\tau}_0}{\eta \nu}, \mathcal{A}_0(\tau_0, \nu, r), \mathcal{A}_0(\tilde{\tau}_0, \nu, r), \mathcal{A}_1(\max(\tau_0, \tilde{\tau}_0) + 1), \mathcal{A}_1\left(\frac{1}{1-2\nu}\right) \right),$$

with \mathcal{A}_0 given by Proposition 2, \mathcal{A}_1 given by Proposition 1. Then, according to Lemma 2 and 3, we can define the two $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible subsequences (Q_l) and (R_l) .

Consider the quasi-periodic Schrödinger cocycle $(\alpha, A_{(E,V)})$:

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{(E,V)}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \text{ with } A_{(E,V)}(\theta) = \begin{pmatrix} -E + V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Its fibred rotation number $\rho = \rho_{(\alpha, A_{(E,V)})} : \mathbb{R} \rightarrow [0, \pi]$ is monotonic. Given $\varepsilon > 0$, we define $\Omega_l, \overline{\Omega}_l$ and $\overline{\Omega}$ for the subsequence (Q_l) and parameters $(\varepsilon, \tau_0, \nu)$ as in the previous section. We can see that Ω_l is a union of at most $2Q_l$ intervals. Let $\Sigma_0 := \rho^{-1}(\mathcal{Q}_\alpha(\varepsilon, \tau, \nu))$. It is clear that $\Sigma_0 \subset \overline{\Omega}$ since, by (5.3), $\tau \leq \tau_0$.

For any E such that the Lyapunov exponent vanishes, it is natural to ask if the associated cocycle is analytically conjugated to constant matrix. As a related concept, “almost reducibility” was introduced by Eliasson[17], and has been developed in [4] and [6] for one-frequency models. Roughly speaking, a cocycle is said to be almost reducible if, by a sequence of transformations, the transformed cocycles are closer and closer to constant.

Definition 4 (almost reducible). *Given $A_0 \in SL(2, \mathbb{R})$, the cocycle $(\alpha, A_{(E,V)})$ is called almost reducible to A_0 , if there exist $h > 0$, a sequence $\{\Lambda_n\}_{n \geq 0} \subset C^\omega(\mathbb{T}, SL(2, \mathbb{R}))$, analytic on \mathbb{T}_h , such that*

$$\lim_{n \rightarrow \infty} |\Lambda_n(\cdot + \alpha)^{-1} A_{(E,V)}(\cdot) \Lambda_n(\cdot) - A_0|_h = 0.$$

By Theorem 1.4 and Corollary 1.6 of [4], we have

LEMMA 8. *For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $V \in C^\omega(\mathbb{T}, \mathbb{R})$ such that L_θ has purely a.c. spectrum for a.e. θ , then for a.e. $E \in \sigma(L)$, the cocycle $(\alpha, A_{(E,V)})$ is almost reducible to some $A_0 \in SO(2, \mathbb{R})$.*

For more on reducibility or almost reducibility for $SL(2, \mathbb{R})$ and $sl(2, \mathbb{R})$ cocycles, and the development of KAM theory in this context, we can refer to [16, 18, 20, 31].

According to Lemma 8, for any E_0 in a co-null set of Σ_0 , there exists $h = h(E_0)$ such that we can find $\Lambda \in C^\omega(\mathbb{T}, SL(2, \mathbb{R}))$, analytic on \mathbb{T}_h , satisfying

$$|\Lambda(\cdot + \alpha)^{-1} A_{(E_0,V)}(\cdot) \Lambda(\cdot) - A_0|_h < \frac{\epsilon_0}{2},$$

where $\epsilon_0 = \epsilon_0(\tau, \nu, \varepsilon, h, \mathcal{A})$ is given in Proposition 2. Then there exists an interval J containing E_0 such that for every $E \in J$,

$$|\Lambda(\cdot + \alpha)^{-1} A_{(E,V)}(\cdot) \Lambda(\cdot) - A_0|_h < \epsilon_0.$$

With the $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible subsequence (Q_l) given in the beginning of this section, by applying Proposition 2 to the cocycle $(\alpha, \Lambda(\cdot + \alpha)^{-1} A_{(\cdot,V)}(\cdot) \Lambda(\cdot))$ parametrised by $E \in J$, we can find

$$\begin{cases} W : (J \cap \overline{\Omega}) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ \phi : (J \cap \overline{\Omega}) \times \mathbb{T} \rightarrow \mathbb{T} \end{cases}, \text{ such that}$$

$$A_{(E,V)} = \Lambda(\cdot + \alpha)W(E, \cdot + \alpha)R_{\phi(E, \cdot)}W(E, \cdot)^{-1}\Lambda^{-1}, \quad \forall E \in J \cap \overline{\Omega}.$$

Thus for every $E \in J \cap \overline{\Omega}$, the corresponding generalized eigenvector is in $\ell^\infty(\mathbb{Z})$.

For any $\theta \in \mathbb{T}_{ac}$, we define $e_0 = e_0(\theta, u(0))$ as the Lebesgue measure of the subset

$$(5.5) \quad \left\{ E \in \sigma(L) \left| \begin{array}{l} E \in \rho^{-1}(\mathcal{Q}_\alpha(\varepsilon, \tau, \nu) = \Sigma_0, \text{ and there exists a generalised eigenvector} \\ g(E) \in \ell^\infty(\mathbb{Z}) \text{ of } L_\theta \text{ such that } \sum_n u_n(0)g_n(E) \neq 0 \end{array} \right. \right\}.$$

By choosing ε small, combining with the absolute continuity of spectrum, we can ensure that $e_0 > 0$.

Note that E_0 is chosen arbitrarily in a co-null set of Σ_0 , we can find finitely many open intervals $\{J_k\}_{1 \leq k \leq K}$ with $\{cl\{J_k\}\}$ mutually disjoint and $|\Sigma_0 \setminus \cup_k J_k| < \frac{e_0}{4}$, such that there is a constant $h > 0$

$$\text{and } \begin{cases} \Lambda : (\cup_k J_k) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \\ A_0 : \cup_k J_k \rightarrow SO(2, \mathbb{R}) \end{cases} \text{ satisfying}$$

- both of Λ and A_0 are constant on each J_k ;
- Λ is analytic on \mathbb{T}_h for any $E \in \cup_k J_k$;
- $|\Lambda(\cdot + \alpha)^{-1}A_{(E,V)}(\cdot)\Lambda - A_0|_{\cup_k J_k, h} < \varepsilon_0$.

Indeed, we can choose a compact set $\Sigma_1 \subset \Sigma_0$ such that Σ_1 contains only almost reducible energy and $|\Sigma_0 \setminus \Sigma_1| < \frac{e_0}{8}$. For each $E_0 \in \Sigma_1$, we choose the interval J containing E_0 described as above. Then by compactness, we can choose finitely many such intervals to cover Σ_1 . Then we choose a disjoint finer covering and choose $A_0(E, \cdot)$, $\Lambda(E, \cdot)$ according to which interval E originally belongs to. Then we adjust h accordingly.

Apply Proposition 2(1) to $\Lambda(\cdot + \alpha)^{-1}A_{(\cdot, V)}\Lambda$ on each J_k , we get W and ϕ . Then we can define

$$Z : ((\cup_k J_k) \cap \overline{\Omega}) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \text{ with } Z(E, \cdot) := \Lambda(\cdot)W(E, \cdot).$$

By Proposition 2(2), we get the approximants W_l and ϕ_l . So we also define

$$Z_l : ((\cup_k J_k) \cap \overline{\Omega}_l) \times \mathbb{T} \rightarrow SL(2, \mathbb{R}) \text{ with } Z_l(E, \cdot) := \Lambda(\cdot)W_l(E, \cdot).$$

By the monotonicity of the Schrödinger operators, it is direct to check that the condition (4.5) in Proposition 2(3) is satisfied with Λ and $A(E, \cdot) = \Lambda(\cdot + \alpha)^{-1}A_{(E,V)}\Lambda$ given as above with $m_1 = 2$, thus we can find $n_0, l_0 \in \mathbb{N}$, $C_1 > 0$ such that for $n > n_0, l > l_0, 1 \leq k \leq K$

$$(5.6) \quad \partial_E \phi_l^{[n]}(E, \theta) > C_1 n, \quad \forall E \in J_k \cap \overline{\Omega}_l, \theta \in \mathbb{T}.$$

Moreover there exists $c_1 > 0$, for any $l \geq 1$, any $n \in \mathbb{Z}$, any $1 \leq k \leq K$,

$$(5.7) \quad \left| \phi_l^{[n]} \right|_{C^r(J_k \cap \overline{\Omega}_l)} \leq c_1 |n|.$$

REMARK 5. We have $(\cup_k J_k) \cap \overline{\Omega} \subset \sigma(L)$ by the (rotational) reducibility.

REMARK 6. Recall that we have constructed another $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible sequence (R_l) in Lemma 3. We can define the subsets $\Gamma_l, \overline{\Gamma}_l$ and $\overline{\Gamma}$ for (R_l) and parameters $(\varepsilon, \tilde{\tau}_0, \nu)$ in the same way as $\Omega_l, \overline{\Omega}_l$ and $\overline{\Omega}$. We have $\Sigma_0 \subset \overline{\Gamma}$ since by (5.3) $\tau \leq \tilde{\tau}_0$. Then we can apply Proposition 2 to (R_l) on $J_k \cap \overline{\Gamma}$ (and $J_k \cap \overline{\Gamma}_l$), and get the conjugation matrix \tilde{W} , angle function $\tilde{\phi}$ and their approximants $\tilde{W}_l, \tilde{\phi}_l$. So we can define $\tilde{Z}(E, \cdot) := \Lambda(\cdot)\tilde{W}(E, \cdot), \tilde{Z}_l(E, \cdot) := \Lambda(\cdot)\tilde{W}_l(E, \cdot)$.

The following measure estimate is based on the estimates obtained in Proposition 2 and the monotonic property of Schrödinger cocycles.

LEMMA 9. *There exists $C_4 > 0$ such that for sufficiently large l , we have*

$$|(\cup_k J_k) \cap (\overline{\Omega}_l \setminus \overline{\Omega})| \leq C_4 \max(Q_{l+1}^{-\tau_0}, \overline{Q}_{l+1}^{-\nu}).$$

Proof. Recall that by applying Proposition 2 to $\Lambda(\cdot + \alpha)^{-1} A_{(E,V)}(\cdot) \Lambda(\cdot)$ on each J_k , with parameters ε, τ_0, ν , we obtain W_m, ϕ_m as above with

$$A_m(E, \cdot) = W_m(E, \cdot + \alpha)^{-1} \Lambda(\cdot + \alpha)^{-1} A_{(E,V)}(\cdot) W_m(E, \cdot) = R_{\phi_m(E, \cdot)}(id + \xi_m(E, \cdot)).$$

By Claim 4.6 in [6], we have, for any $\theta \in \mathbb{T}$, and $1 \leq l \leq Q_{m+1}$,

$$A_m^{(l)}(E, \theta) = R_{\phi_m(E, \theta + (l-1)\alpha)}(id + \xi_m(E, \theta + (l-1)\alpha)) \cdots R_{\phi_m(E, \theta)}(id + \xi_m(E, \theta))$$

can be expressed as $R_{\phi_m^{[l]}(E, \theta)}(id + \xi^{(l)}(E, \theta))$ such that for $1 \leq l \leq Q_{m+1}$,

$$|\xi^{(l)}(E, \cdot)|_{J \cap \overline{\Omega}_l, h_l} \leq \exp \left\{ \sum_{k=0}^{l-1} |\xi_m(E, \cdot + k\alpha)| \right\} - 1 < 10c_1 Q_{m+1} e^{-\overline{Q}_m^{D_1}}$$

For $l = Q_{m+1}$, by Proposition 1 (apply to $\eta = 1, h = h_l > h_\infty$) and (5.4), (5.7), when m is sufficiently large, $C = C(h_\infty, 1/2, \mathcal{A})$ in Proposition 1

$$\left| \phi_m^{[Q_{m+1}]} - Q_{m+1} \hat{\phi}_m(0) \right|_{J_k \cap \overline{\Omega}_m} < C \cdot c_1 (Q_{m+1}^{-\tau_0-1}, \overline{Q}_{m+1}^{-2\nu}) < \frac{\varepsilon}{100} \max(Q_{m+1}^{-\tau_0}, \overline{Q}_{m+1}^{-\nu}).$$

Note that the fibered rotation number of the cocycle $(Q_{k+1}\alpha, A^{(Q_{k+1})})$ is $Q_{k+1}\rho$. Then for each $E \in J_k$, there exists $\theta \in \mathbb{R}$ such that

$$\widehat{A_m^{[Q_{m+1}]}}(E, \theta) - \theta = Q_{m+1}\rho(E),$$

where $\widehat{A_m^{[Q_{m+1}]}}(E, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ denotes a lift of the projective map $A_m^{[Q_{m+1}]}(E, \cdot) : \mathbb{T} \rightarrow \mathbb{T}$. Hence

$$\left| \phi_m^{[Q_{m+1}]} - Q_{m+1}\rho \right|_{J_k \cap \overline{\Omega}_m} \leq \frac{\varepsilon}{10} \max(Q_{m+1}^{-\tau_0}, \overline{Q}_{m+1}^{-\nu}) + 10c_1 Q_{m+1} e^{-\overline{Q}_m^{D_1}}.$$

Thus $J_k \cap (\overline{\Omega}_m \setminus \overline{\Omega}_{m+1}) \subset \{E \in J_k \cap \overline{\Omega}_m \mid \|2\phi_m^{[Q_{m+1}]}(E, \cdot)\| \leq 2\varepsilon \max(Q_{m+1}^{-\tau_0}, \overline{Q}_{m+1}^{-\nu})\}$. Moreover by Proposition 2(3), we have $c_1 Q_{m+1} \geq \partial_E \phi_m^{[Q_{m+1}]}(E, \cdot) > C_1 Q_{m+1}$ for $E \in J_k \cap \overline{\Omega}_m$ for any sufficiently large m . Hence

$$|(\cup_k J_k) \cap (\overline{\Omega}_m \setminus \overline{\Omega}_{m+1})| \leq \frac{c_1}{2C_1} \max(Q_{m+1}^{-\tau_0}, \overline{Q}_{m+1}^{-\nu}).$$

Thus $|(\cup_k J_k) \cap (\overline{\Omega}_l \setminus \overline{\Omega})| \leq C_4 \max(Q_{l+1}^{-\tau_0}, \overline{Q}_{l+1}^{-\nu})$ for $C_4 = \frac{c_1}{C_1}$ when l is sufficiently large. \square

5.3. A measure supported on the spectrum with smoothing properties.

PROPOSITION 3. *There is a numerical constant $c_2 > 0$, and a sequence of C^∞ functions $\{\psi_{(l)}\}_{l \geq 1}$ satisfying that $0 \leq \psi_{(l)} \leq 1$ and $|\psi_{(l)}|_{C^2} \leq c_2 Q_l^{\tau_1}$, such that $B(\text{supp}(\psi_{(1)}), Q_1^{-\tau_0}) \subset \overline{\Omega}_1$, and for $l \geq 2$, with $\psi_l := \prod_{k=1}^l \psi_{(k)}$,*

$$(5.8) \quad B(\text{supp}(\psi_{(l)}), Q_l^{-\tau_0}) \subset \overline{\Omega}_l \cap \text{supp}(\psi_{l-1}), \quad \left| \{E \in \overline{\Omega}_l \cap \text{supp}(\psi_{l-1}) \mid \psi_{(l)}(E) \neq 1\} \right| \leq c_2 Q_l^{-\frac{\tau_1}{4}}.$$

Moreover, with $\psi := \prod_{l=1}^\infty \psi_{(l)}$, we have $\text{supp}(\psi) \subset \overline{\Omega}$ and

$$(5.9) \quad |\psi - \psi_l|_{L^1((\cup_k J_k) \cap \overline{\Omega})} \leq c_2 (Q_{l+1}^{-\frac{\tau_1}{4}} + \overline{Q}_{l+1}^{-\frac{\nu}{2}}).$$

Proof. At first, we focus on these connected components of $\overline{\Omega}_1 = \Omega_1$. Since Ω_1 is a union of at most $2Q_1$ intervals, for those intervals of length $< 4Q_1^{-\frac{\tau_1}{2}}$, the measure of their union is less than

$$8Q_1^{1-\frac{\tau_1}{2}} < \frac{1}{2}Q_1^{-\frac{\tau_1}{4}}.$$

On each interval of length $\geq 4Q_1^{-\frac{\tau_1}{2}}$, denoted by J , we can construct a C^∞ function $\psi_J^{(1)}$ supported on J , satisfying $0 \leq \psi_J^{(1)} \leq 1$ and $|\psi_J^{(1)}|_{C^2} \leq c_2 Q_1^{\tau_1}$, such that

$$B\left(\text{supp}(\psi_J^{(1)}), Q_1^{-\tau_0}\right) \subset J, \quad \left|\{E \in J \mid \psi_J^{(1)}(E) \neq 1\}\right| \leq 4Q_1^{-\frac{\tau_1}{2}}.$$

Note that here we have used the relation $\tau_0 \geq \tau_1$. Let $\psi_{(1)} := \sum_J \psi_J^{(1)}$ where J runs over all the connected components of length $\geq 4Q_1^{-\frac{\tau_1}{2}}$.

Assume that we have already constructed $\psi_{(1)}, \dots, \psi_{(l)}$ satisfying the above properties in (5.8). Moreover, we assume that for each $1 \leq k \leq l$, $\text{supp}(\psi_{(k)})$ is a union of at most $2kQ_k$ intervals. Then we can see that $\text{supp}(\psi_l)$ has at most $2l^2Q_l$ connected components. Since Ω_{l+1} is a union of at most $2Q_{l+1}$ intervals, we can see $\overline{\Omega}_{l+1} \cap \text{supp}(\psi_l)$ has at most $2(l+1)Q_{l+1}$ connected components by noting that $Q_{l+1} > lQ_l$.

We focus on these connected components of $\overline{\Omega}_{l+1} \cap \text{supp}(\psi_l)$. For those intervals of length $< 4Q_{l+1}^{-\frac{\tau_1}{2}}$, the measure of their union is less than $8(l+1)Q_{l+1}^{1-\frac{\tau_1}{2}} < \frac{1}{2}Q_{l+1}^{-\frac{\tau_1}{4}}$. On each interval of length $\geq 4Q_{l+1}^{-\frac{\tau_1}{2}}$, saying J , we can construct a C^∞ function $\psi_J^{(l+1)}$ supported on J , satisfying $0 \leq \psi_J^{(l+1)} \leq 1$ and $|\psi_J^{(l+1)}|_{C^2} \leq c_2 Q_{l+1}^{\tau_1}$, such that

$$B\left(\text{supp}(\psi_J^{(l+1)}), Q_{l+1}^{-\tau_0}\right) \subset J, \quad \left|\{E \in J \mid \psi_J^{(l+1)}(E) \neq 1\}\right| \leq 4Q_{l+1}^{-\frac{\tau_1}{2}}.$$

Let $\psi_{(l+1)} := \sum_J \psi_J^{(l+1)}$ where J runs over all the connected components of length $\geq 4Q_{l+1}^{-\frac{\tau_1}{2}}$. It is direct to check that ψ_{l+1} satisfies the induction assumption.

Since for each m , we have $\left|\{E \in \overline{\Omega}_m \cap \text{supp}(\psi_{m-1}) \mid \psi_{(m)}(E) \neq 1\}\right| \lesssim Q_m^{-\frac{\tau_1}{4}}$. Combined with Lemma 9, (5.9) follows from the inclusion

$$\{E : \psi(E) \neq \psi_l(E)\} \subset (\overline{\Omega}_l \setminus \overline{\Omega}) \cup \left(\bigcup_{m=l+1}^\infty \{E \in \overline{\Omega}_m \cap \text{supp}(\psi_{m-1}) \mid \psi_{(m)}(E) \neq 1\}\right).$$

□

It is direct to see that we have $|\psi_l|_{C^2(\overline{\Omega}_l)} \lesssim l^2 Q_l^{\tau_1}$ and

$$\text{supp}(\psi) \subset \overline{\Omega}, \quad |\{E \in \overline{\Omega} \mid \psi(E) = 0\}| \lesssim \sum_{l=1}^\infty Q_l^{-\frac{\tau_1}{4}} \lesssim Q_1^{-\frac{\tau_1}{4}}.$$

Recall that $\Sigma_0 = \rho^{-1}(\mathcal{Q}_\alpha(\varepsilon, \tau, \nu)) \subset \overline{\Omega}$. After possibly modifying $\psi_{(l)}$ for finitely many index l (at the cost of enlarging the constant c_2 in Proposition 3), we can assume that

$$(5.10) \quad \left| \left\{ E \in (\cup_k J_k) \cap \Sigma_0 \mid \begin{array}{l} \psi(E) > 0, \exists \text{ a } \ell^\infty \text{ generalised eigenvector } g(E) \\ \text{of } L \text{ such that } \sum_n u_n(0)g_n(E) \neq 0 \end{array} \right\} \right| > \frac{e_0}{2},$$

recalling that e_0 is the Lebesgue measure of the subset given in (5.5).

REMARK 7. The function ψ will serve as a smoothing measure for the modified spectral transformation. Similarly as in Remark 6, with (Q_l) replaced by (R_l) , we can construct functions $\tilde{\psi}_{(l)}$, $\tilde{\psi}_l$ and $\tilde{\psi}$ in the same way.

The following lemma shows that Z, ϕ are differentiable in the sense of Whitney on the support of ψ . We follow the inductive step to find their extensions.

LEMMA 10. Z and ϕ are C^{r-2} in the sense of Whitney with respect to E on $(\cup_k J_k) \cap \text{supp}(\psi)$. Moreover, there exist $c_3, D_2 > 0$ such that

$$|\partial_E^j(Z - Z_l)|_{(\cup_k J_k) \cap \text{supp}(\psi)}, |\partial_E^j(\phi - \phi_l)|_{(\cup_k J_k) \cap \text{supp}(\psi)} < c_3 e^{-\overline{Q}_l^{D_2}}, \quad j = 0, \dots, r-2$$

Proof. Fix any $\theta \in \mathbb{T}$. We only detail the proof for Z , with that of ϕ being similar.

Let $F_1 = Z_1$. By (5.8), there exists a sequence of C^∞ functions on \mathbb{R} , denoted by $\{\Psi_l\}_l$, with $0 \leq \Psi_l \leq 1$ and $|\Psi_l|_{C^{r-1}(\mathbb{R})} \lesssim Q_l^{(r-1)\tau_0}$ for each $l \geq 1$, such that

$$(5.11) \quad \text{supp}(\Psi_{l+1}) \subset \text{supp}(\psi_l) \cap \overline{\Omega}_{l+1} \subset \Psi_l^{-1}(1).$$

For $l \geq 2$, we define $F_l := \Psi_l Z_l + (1 - \Psi_l)F_{l-1}$. Hence, $F_{l+1} - F_l = \Psi_{l+1}(Z_{l+1} - F_l)$, and $F_l(E) = Z_l(E)$ for all $E \in \Psi_l^{-1}(1)$. So $F_{l+1} - F_l = \Psi_{l+1}(Z_{l+1} - Z_l)$ on $\Psi_l^{-1}(1)$.

Recall that $Z_l = \Lambda W_l$. On each J_k , $1 \leq k \leq K$, by Proposition 2 and (5.11),

$$|F_{l+1} - F_l|_{C^{r-1}(J_k \cap \overline{\Omega}_0)} \leq |\Lambda| |\Psi_{l+1}|_{C^{r-1}(\mathbb{R})} |W_{l+1} - W_l|_{C^{r-1}(J_k \cap \overline{\Omega}_{l+1})} \lesssim c_1 |\Lambda| Q_{l+1}^{(r-1)\tau_0} e^{-\overline{Q}_l^{D_1}}.$$

Thus $\{F_l\}$ is a Cauchy sequence in $C^{r-1}(J_k \cap \overline{\Omega}_0)$. So there exists $F \in C^{r-2}((\cup_k J_k) \cap \overline{\Omega}_0)$ such that $\lim_{l \rightarrow \infty} |F - F_l|_{C^{r-2}((\cup_k J_k) \cap \overline{\Omega}_0)} = 0$. Moreover, there exist constants $c_3, D_2 > 0$, independent of l , such that

$$|\partial_E^j(F_l - F)|_{(\cup_k J_k) \cap \overline{\Omega}_0} < c_3 e^{-\overline{Q}_l^{D_2}}, \quad j = 0, \dots, r-2.$$

On the other hand, since $F_l = Z_l$ on $(\cup_k J_k) \cap \Psi_l^{-1}(1)$, combining with the fact that $\text{supp}(\psi_l) \cap \overline{\Omega}_{l+1} \subset \Psi_l^{-1}(1)$, we conclude that F_l converges to Z in C^0 on $(\cup_k J_k) \cap \text{supp}(\psi)$, noting that $\text{supp}(\psi) \subset \overline{\Omega}$, which follows from Proposition 3. Thus $F = Z$ on $(\cup_k J_k) \cap \text{supp}(\psi)$. This concludes the proof since F is C^{r-2} over $(\cup_k J_k) \cap \overline{\Omega}_0$. \square

5.4. Modified spectral transformation. For each $1 \leq k \leq K$, we choose a non-negative real function $\chi_k \in C^\infty(\mathbb{R})$ such that $\text{supp}(\chi_k) = cl(J_k)$, i.e. $\chi_k(E) > 0$ for any $E \in J_k$ and $\chi_k(E) = 0$ for any $E \notin J_k$. Obviously, χ_k vanishes on ∂J_k . We further require that $\partial_E^i \chi_k(E) = 0$ for $E \in \partial J_k$ for $i = 1, 2, 3$. We define $\chi \in C^\infty(\mathbb{R})$ by $\chi = \sum_k \chi_k$.

Since $A_{(E,V)}(\theta) = Z(E, \theta + \alpha) R_{\phi(E,\theta)} Z(E, \theta)^{-1}$ on $((\cup_k J_k) \cap \overline{\Omega}) \times \mathbb{T}$, we can see that $L_\theta f = Ef$ with

$$f_n(E, \theta) := e^{i\phi^{[n]}(E,\theta)} [Z_{21}(E, \theta + n\alpha) - iZ_{22}(E, \theta + n\alpha)].$$

Indeed, by noting that $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector of $R_{\phi(E,\theta)}$ corresponding to the eigenvalue $e^{i\phi(E,\theta)}$,

with $\begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = Z(E, \theta) \begin{pmatrix} 1 \\ -i \end{pmatrix}$, we get the generalized eigenvector

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = Z(E, \theta + n\alpha) R_{\phi^{[n]}(E,\theta)} Z(E, \theta)^{-1} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = e^{i\phi^{[n]}(E,\theta)} Z(E, \theta + n\alpha) \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Then, for every $n \in \mathbb{Z}$, let

$$\begin{pmatrix} \mathcal{K}_n \\ \mathcal{J}_n \end{pmatrix} = \chi \begin{pmatrix} \Im f_n \\ \Re f_n \end{pmatrix} = \begin{pmatrix} \beta_n \sin \phi^{[n]} - \gamma_n \cos \phi^{[n]} \\ \beta_n \cos \phi^{[n]} + \gamma_n \sin \phi^{[n]} \end{pmatrix},$$

where $\beta_n(E, \theta) := \chi(E)Z_{21}(E, \theta + n\alpha)$, $\gamma_n(E, \theta) := \chi(E)Z_{22}(E, \theta + n\alpha)$. Obviously, $\mathcal{K} = (\mathcal{K}_n)_{n \in \mathbb{Z}}$ and $\mathcal{J} = (\mathcal{J}_n)_{n \in \mathbb{Z}}$ are two generalized eigenvectors of L . Moreover, for every $l \in \mathbb{N}_+$, we can define the approximated coefficients

$$\beta_n^{(l)}(E, \theta) := \chi(E)(Z_l)_{21}(E, \theta + n\alpha), \quad \gamma_n^{(l)}(E, \theta) := \chi(E)(Z_l)_{22}(E, \theta + n\alpha).$$

LEMMA 11. For each $1 \leq k \leq K$, β_n, γ_n are C^{r-2} in the sense of Whitney on $J_k \cap \overline{\Omega}$, and $\beta_n^{(l)}, \gamma_n^{(l)}$ are C^∞ on $J_k \cap \overline{\Omega}_l$. Moreover, there exists $c_4 > 0$ such that, for every n and every l, k , we have

$$(5.12) \quad |\beta_n|_{C^{r-2}(J_k \cap \overline{\Omega})}, |\gamma_n|_{C^{r-2}(J_k \cap \overline{\Omega})}, |\beta_n^{(l)}|_{C^r(J_k \cap \overline{\Omega}_l)}, |\gamma_n^{(l)}|_{C^r(J_k \cap \overline{\Omega}_l)} \leq c_4,$$

$$(5.13) \quad |\partial_E^j(\beta_n - \beta_n^{(l)})|_{J_k \cap \text{supp}(\psi)}, |\partial_E^j(\gamma_n - \gamma_n^{(l)})|_{J_k \cap \text{supp}(\psi)} \leq c_4 e^{-\overline{Q}_l^{D_2}}, \quad j = 0, \dots, r-2,$$

Proof. The regularity of $\beta_n, \gamma_n, \beta_n^{(l)}, \gamma_n^{(l)}$ follows easily from their definitions.

Recall that $\beta_n = \chi Z_{21}(\theta + n\alpha)$, $\beta_n^{(l)} = \chi(Z_l)_{21}(\theta + n\alpha)$ with $Z = \Lambda W$, $Z_l = \Lambda W_l$. Then, by (4.2) in Proposition 2, we have

$$|\beta_n|_{C^{r-2}(J_k \cap \overline{\Omega})}, |\beta_n^{(l)}|_{C^r(J_k \cap \overline{\Omega}_l)} \leq 6c_1 |\chi|_{C^r(J_k)} |\Lambda|.$$

and in view of Lemma 10, we have

$$|\partial_E^j(\beta_n - \beta_n^{(l)})|_{J_k \cap \text{supp}(\psi)} \leq 2c_3 |\chi|_{C^{r-2}(J_k)} e^{-\overline{Q}_l^{D_2}}, \quad j = 0, \dots, r-2.$$

By choosing $c_4 := 10 \max(c_1 |\Lambda|, c_3) \cdot |\chi|_{C^r(J_k)}$, we get the estimates (5.12), (5.13) for β_n and $\beta_n^{(l)}$. The proof for γ_n and $\gamma_n^{(l)}$ is similar. \square

Let $d\mu := \begin{pmatrix} \psi dE & 0 \\ 0 & \psi dE \end{pmatrix}$. Recall the definition of \mathcal{L}^2 -space given in (2.2). $\mathcal{L}^2(d\mu)$ means the space of vectors $G = (g_j)_{j=1,2}$, with g_j functions of $E \in \mathbb{R}$ satisfying

$$\|G\|_{\mathcal{L}^2(d\mu)}^2 := \int_{\mathbb{R}} (|g_1|^2 + |g_2|^2) \psi dE < \infty.$$

For any $\theta \in \mathbb{T}_{ac}$, we define the modified spectral transformation for the operator L_θ by

$$\begin{aligned} \mathcal{S}_\theta : \mathcal{W}^2(\mathbb{Z}) &\rightarrow \mathcal{L}^2(d\mu) \\ (u_n)_{n \in \mathbb{Z}} &\mapsto \begin{pmatrix} \sum_{n \in \mathbb{Z}} u_n \mathcal{K}_n(\cdot, \theta) \\ \sum_{n \in \mathbb{Z}} u_n \mathcal{J}_n(\cdot, \theta) \end{pmatrix}. \end{aligned}$$

Recall that $\Sigma_0 \subset \overline{\Omega}$. Since $\mathcal{K}_n, \mathcal{J}_n$ are uniformly bounded for all $(E, \theta) \in ((\cup_k J_k) \cap \Sigma_0) \times \mathbb{T}$ and $n \in \mathbb{Z}$, and $0 \leq \psi \leq 1$, we can see by Cauchy inequality that for any $u \in \mathcal{W}^2(\mathbb{Z})$,

$$\left\| \begin{pmatrix} \sum_{n \in \mathbb{Z}} u_n \mathcal{K}_n(\cdot, \theta) \\ \sum_{n \in \mathbb{Z}} u_n \mathcal{J}_n(\cdot, \theta) \end{pmatrix} \right\|_{\mathcal{L}^2(d\mu)}^2 \leq (\|\mathcal{K}\|_{\ell^\infty}^2 + \|\mathcal{J}\|_{\ell^\infty}^2) \sum_{m, n \in \mathbb{Z}} |u_m| |u_n| \lesssim \langle u \rangle_2^2.$$

For $M > 0$, we also define the truncated spectral transformation

$$\begin{aligned} \mathcal{S}_{\theta, M} : \ell^2(\mathbb{Z}) &\rightarrow \mathcal{L}^2(d\mu) \\ (u_n)_{n \in \mathbb{Z}} &\mapsto \begin{pmatrix} \sum_{|n| \leq M} u_n \mathcal{K}_n(\cdot, \theta) \\ \sum_{|n| \leq M} u_n \mathcal{J}_n(\cdot, \theta) \end{pmatrix}. \end{aligned}$$

It is clear that given any $u \in \mathcal{W}^2(\mathbb{Z})$, for every $E \in ((\cup_k J_k) \cap \overline{\Omega})$,

$$\lim_{M \rightarrow \infty} \sum_{|n| > M} |u_n \mathcal{K}_n(E, \theta)| = \lim_{M \rightarrow \infty} \sum_{|n| > M} |u_n \mathcal{J}_n(E, \theta)| = 0.$$

So $S_\theta u$ is the pointwise limit of $S_{\theta,M} u$ as $M \rightarrow \infty$.

For given $\theta \in \mathbb{T}_{ac}$, we choose $\varepsilon > 0$ and ψ accordingly, so that by (5.10) we have

$$\left| \left\{ E \in (\cup_k J_k) \cap \Sigma_0 \mid \psi(E) > 0, \left(\frac{\sum_n u_n(0) \mathcal{K}_n(E, \theta)}{\sum_n u_n(0) \mathcal{J}_n(E, \theta)} \right) \neq 0 \right\} \right| > \frac{e_0}{2}.$$

Then, as the modified spectral transformation with measure $d\mu$, S_θ has the property that $S_\theta u \neq 0$. Thus by dominated convergence theorem, we have

$$(5.14) \quad \lim_{M \rightarrow \infty} \|S_{\theta,M} u\|_{\mathcal{L}^2(d\mu)} = \|S_\theta u\|_{\mathcal{L}^2(d\mu)} > 0.$$

REMARK 8. As shown in Subsection 2.1, the classical spectral transformation is a unitary transformation from $\ell^2(\mathbb{Z})$ to $\mathcal{L}^2(d\mu)$, with $d\mu$ the matrix of spectral measures introduced by m -functions. In contrast, to get better differentiability with respect to E , the modified spectral transformation S_θ here is not a unitary one, and we define it just on the subspace $\mathcal{W}^2(\mathbb{Z})$. Comparing with (2.3) for the free Schrödinger operator, \mathcal{K}_n and \mathcal{J}_n for S_θ have no divisor as “ $\sim \sin \xi_0$ ” and they have a smoothing factor χ which covers the singularities. Moreover, instead of the spectral measures shown in Theorem 2, we use the explicit measure ψdE , which serves also as a smoothing factor.

By direct computations, we obtain

$$(5.15) \quad \begin{pmatrix} \partial_E \mathcal{K}_n \\ \partial_E \mathcal{J}_n \end{pmatrix} = \begin{pmatrix} \mathcal{B}_n \cos \phi^{[n]} + \mathcal{C}_n \sin \phi^{[n]} \\ -\mathcal{B}_n \sin \phi^{[n]} + \mathcal{C}_n \cos \phi^{[n]} \end{pmatrix}$$

with $\mathcal{B}_n := \beta_n \partial_E \phi^{[n]} - \partial_E \gamma_n$, $\mathcal{C}_n := \gamma_n \partial_E \phi^{[n]} + \partial_E \beta_n$, where ∂_E is interpreted in the sense of Whitney. Similarly, we define the approximated coefficients $\mathcal{B}_n^{(l)} := \beta_n^{(l)} \partial_E \phi_l^{[n]} - \partial_E \gamma_n^{(l)}$, $\mathcal{C}_n^{(l)} := \gamma_n^{(l)} \partial_E \phi_l^{[n]} + \partial_E \beta_n^{(l)}$. According to Lemma 10 and 11, β_n, γ_n are bounded on $J_k \cap \text{supp}(\psi)$, and $\beta_n^{(l)}, \gamma_n^{(l)}$ are C^∞ on $J_k \cap \overline{\Omega}_l$. Combining with the fact that $|\partial_E \phi_l^{[n]}| \leq c_1 n$, we can find a constant $c_5 > 0$, such that

$$(5.16) \quad |\mathcal{B}_n|_{J_k \cap \text{supp}(\psi)}, |\mathcal{C}_n|_{J_k \cap \text{supp}(\psi)}, |\mathcal{B}_n^{(l)}|_{C^2(J_k \cap \overline{\Omega}_l)}, |\mathcal{C}_n^{(l)}|_{C^2(J_k \cap \overline{\Omega}_l)} \leq c_5 n,$$

$$(5.17) \quad |\mathcal{B}_n - \mathcal{B}_n^{(l)}|_{J_k \cap \text{supp}(\psi)}, |\mathcal{C}_n - \mathcal{C}_n^{(l)}|_{J_k \cap \text{supp}(\psi)} \leq c_5 n e^{-\overline{Q}_l^{D_2}},$$

$$(5.18) \quad |\partial_E \mathcal{K}_n|_{J_k \cap \text{supp}(\psi)}, |\partial_E \mathcal{J}_n|_{J_k \cap \text{supp}(\psi)} \leq c_5 n.$$

To consider the ballistic lower bound for any $p \geq 0$, it can be reduced to considering the case that p is an even integer. This will be discussed in Subsection 5.6. Assume that $p \geq 2$ is an even integer, and let $r = p/2 + 2$. We have

$$(5.19) \quad \begin{pmatrix} \partial_E^{r-2} \mathcal{K}_n \\ \partial_E^{r-2} \mathcal{J}_n \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{n,r} \cos \phi^{[n]} + \mathcal{C}_{n,r} \sin \phi^{[n]} \\ -\mathcal{B}_{n,r} \sin \phi^{[n]} + \mathcal{C}_{n,r} \cos \phi^{[n]} \end{pmatrix}$$

with $\mathcal{B}_{n,r}, \mathcal{C}_{n,r}$ being a linear combination of monomials of $\partial_E^j \beta_n, \partial_E^j \gamma_n, \partial_E^j \phi^{[n]}$ with sum of the degrees of derivatives no more than $r - 2$. We define the approximants $\mathcal{B}_{n,r}^{(l)}, \mathcal{C}_{n,r}^{(l)}$ in the similar way. By Lemma 10, 11 and (4.3), there exists $c_6 > 0$ such that

$$(5.20) \quad |\mathcal{B}_{n,r}|_{J_k \cap \text{supp}(\psi)}, |\mathcal{C}_{n,r}|_{J_k \cap \text{supp}(\psi)}, |\mathcal{B}_{n,r}^{(l)}|_{C^2(J_k \cap \overline{\Omega}_l)}, |\mathcal{C}_{n,r}^{(l)}|_{C^2(J_k \cap \overline{\Omega}_l)} \leq c_6 n^{r-2},$$

$$(5.21) \quad |\mathcal{B}_{n,r} - \mathcal{B}_{n,r}^{(l)}|_{J_k \cap \text{supp}(\psi)}, |\mathcal{C}_{n,r} - \mathcal{C}_{n,r}^{(l)}|_{J_k \cap \text{supp}(\psi)} \leq c_6 n^{r-2} e^{-\overline{Q}_l^{D_2}},$$

$$(5.22) \quad |\partial_E^{r-2} \mathcal{K}_n|_{J_k \cap \text{supp}(\psi)}, |\partial_E^{r-2} \mathcal{J}_n|_{J_k \cap \text{supp}(\psi)} \leq c_6 n^{r-2}.$$

5.5. Ballistic lower bound for $p = 2$. In this subsection, we will prove Theorem 1 for $p = 2$. This corresponds to the estimates for $r = 3$. The proof for the general case for arbitrary p is completely analogy to that of $p = 2$, we will sketch the needed modifications in the next subsection.

From now on, we fix $\theta \in \mathbb{T}_{ac}$, and we will not express this dependence explicitly. With $u(t)$ the solution of Eq. (5.1) with initial condition $u(0) = u \in \mathcal{W}^2(\mathbb{Z}) \setminus \{0\}$, we consider

$$\begin{aligned} G(t, E) &= \left(G^{(j)}(t, E) \right)_{j=1,2} := S_\theta(u(t)), \\ G_T(t, E) &= \left(G_T^{(j)}(t, E) \right)_{j=1,2} := S_{\theta, \lfloor T^{100} \rfloor + 1}(u(t)) \text{ for any } T > 0. \end{aligned}$$

According to Eq. (5.1), we have

$$\begin{aligned} i\partial_t G_T(t, E) &= i \begin{pmatrix} \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} \partial_t u_n(t) \mathcal{K}_n(E, \theta) \\ \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} \partial_t u_n(t) \mathcal{J}_n(E, \theta) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} (Lu)_n(t) \mathcal{K}_n(E, \theta) \\ \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} (Lu)_n(t) \mathcal{J}_n(E, \theta) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} u_n(t) (L\mathcal{K})_n(E, \theta) \\ \sum_{|n| \leq \lfloor T^{100} \rfloor + 1} u_n(t) (L\mathcal{J})_n(E, \theta) \end{pmatrix} + F_T(t, E) \\ &= EG_T(t, E) + F_T(t, E), \end{aligned}$$

where $F_T(t, E) = \left(F_T^{(j)}(t, E) \right)_{j=1,2}$ is given by

$$\begin{aligned} F_T(t, E) &= \begin{pmatrix} u_{\lfloor T^{100} \rfloor + 1}(t) \mathcal{K}_{\lfloor T^{100} \rfloor + 2}(E, \theta) - u_{\lfloor T^{100} \rfloor + 2}(t) \mathcal{K}_{\lfloor T^{100} \rfloor + 1}(E, \theta) \\ u_{\lfloor T^{100} \rfloor + 1}(t) \mathcal{J}_{\lfloor T^{100} \rfloor + 2}(E, \theta) - u_{\lfloor T^{100} \rfloor + 2}(t) \mathcal{J}_{\lfloor T^{100} \rfloor + 1}(E, \theta) \end{pmatrix} \\ &\quad + \begin{pmatrix} u_{-\lfloor T^{100} \rfloor - 1}(t) \mathcal{K}_{-\lfloor T^{100} \rfloor - 2}(E, \theta) - u_{-\lfloor T^{100} \rfloor - 2}(t) \mathcal{K}_{-\lfloor T^{100} \rfloor - 1}(E, \theta) \\ u_{-\lfloor T^{100} \rfloor - 1}(t) \mathcal{J}_{-\lfloor T^{100} \rfloor - 2}(E, \theta) - u_{-\lfloor T^{100} \rfloor - 2}(t) \mathcal{J}_{-\lfloor T^{100} \rfloor - 1}(E, \theta) \end{pmatrix}. \end{aligned}$$

Hence we have $G_T^{(j)}(t, E) = e^{-itE} G_T^{(j)}(0, E) - i \int_0^t e^{-i(t-s)E} F_T^{(j)}(s, E) ds$, and

$$\begin{aligned} \partial_E G_T^{(j)}(t, E) &= (-it) e^{-itE} G_T^{(j)}(0, E) + e^{-itE} \partial_E G_T^{(j)}(0, E) \\ (5.23) \quad &\quad - \int_0^t (t-s) e^{-i(t-s)E} F_T^{(j)}(s, E) ds - i \int_0^t e^{-i(t-s)E} \partial_E F_T^{(j)}(s, E) ds. \end{aligned}$$

The following two lemmas show that F_T is negligible up to time T .

LEMMA 12. *There is a constant $c_7 > 0$ such that, given any $T > 1$, for $0 \leq t \leq T$,*

$$(5.24) \quad \|F_T(t, E)\|_{\mathcal{L}^2(d\mu)}^2 \leq c_7 T^{-3}.$$

Proof. By the expression of F_T , we have, for $j = 1, 2$,

$$|F_T^{(j)}(t, E)| \leq (\|\mathcal{K}\|_{\ell^\infty} + \|\mathcal{J}\|_{\ell^\infty}) \sum_{i=1,2} \sum_{N=\pm(\lfloor T^{100} \rfloor + i)} |u_N(t)|.$$

For $N = \lfloor T^{100} \rfloor + 1, \lfloor T^{100} \rfloor + 2, u_N(t) = \langle e^{-itL} u(0), \delta_N \rangle = \langle u(0), e^{itL} \delta_N \rangle$. Thus for any $0 \leq t \leq T$,

$$|u_N(t)| \leq \|u(0)\|_{\ell^2(\mathbb{Z})} \left(\sum_{n < T^{10}} |(e^{itL} \delta_N)_n|^2 \right)^{\frac{1}{2}} + \|e^{itL} \delta_N\|_{\ell^2(\mathbb{Z})} \left(\sum_{n \geq T^{10}} |u_n(0)|^2 \right)^{\frac{1}{2}}.$$

By Cauchy inequality,

$$\sum_{n \geq T^{10}} |u_n(0)|^2 \leq \left(\sum_{n \geq T^{10}} \frac{1}{n^2} \right) \langle u(0) \rangle_2^2 \lesssim T^{-10} \langle u(0) \rangle_2^2.$$

Let $S : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the unitary operator defined as $(Sq)_n = q_{n-1}$. Thus

$$\begin{aligned} \sum_{n < T^{10}} |(e^{itL} \delta_N)_n|^2 &= \sum_{n+N < T^{10}} |(e^{itL} \delta_N)_{n+N}|^2 \\ &= \sum_{n < T^{10}-N} |(S^{-N} e^{itL} S^N \delta_0)_n|^2 \\ (5.25) \quad &\leq \left(\sum_{n < T^{10}-N} \frac{1}{n^2} \right) \left(\sum_{n \in \mathbb{Z}} n^2 |(S^{-N} e^{itL} S^N \delta_0)_n|^2 \right). \end{aligned}$$

By the fact that $N \gg 2T^{10}$, we see

$$\sum_{n < T^{10}-N} \frac{1}{n^2} < \sum_{n < -T^{10}} \frac{1}{n^2} \lesssim T^{-10}.$$

Note that $S^{-N} e^{itL} S^N$ equals to $e^{it(S^{-N} L S^N)}$ and the operator $S^{-N} L S^N : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ satisfies

$$(S^{-N} L S^N q)_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\alpha + N\alpha)q_n.$$

Then, applying Theorem 2.22 in [14] with $p = 2$ and $\eta = \frac{3}{2}$, we get, for any $0 \leq t \leq T$,

$$\sum_{n < T^{10}} |(e^{itL} \delta_N)_n|^2 \lesssim T^{-10} \cdot (t^3 + 1) \lesssim T^{-6}.$$

Hence $|u_N(t)| \lesssim \|u(0)\|_{\ell^2(\mathbb{Z})} T^{-3} + \langle u(0) \rangle_2 T^{-5}$ for $N = \lfloor T^{100} \rfloor + 1, \lfloor T^{100} \rfloor + 2$. In the same way, we can get the same estimate for $N = -\lfloor T^{100} \rfloor - 1, -\lfloor T^{100} \rfloor - 2$. So there exists a constant $c_7 = c_7(\langle u(0) \rangle_2, \|u(0)\|_{\ell^2(\mathbb{Z})}, \|\mathcal{K}\|_{\ell^\infty})$ such that for $j = 1, 2$,

$$|F_T^{(j)}(t, E)| \leq \frac{\sqrt{c_7}}{4} T^{-3}, \quad \forall 0 \leq t \leq T.$$

This proves (5.24). □

LEMMA 13. We have $\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \|\partial_E F_T(t, E)\|_{\mathcal{L}^2(d\mu)}^2 = 0$.

Proof. In view of the expression of $\partial_E F_T$, combining with (5.18), we have, for $j = 1, 2$,

$$|\partial_E F_T^{(j)}(t, E)| \lesssim \sum_{i=1,2} \sum_{N=\pm(\lfloor T^{100} \rfloor + i)} |Nu_N(t)|.$$

By proceeding in a similar fashion as in the previous lemma, we obtain, for $N = \lfloor T^{100} \rfloor + 1, \lfloor T^{100} \rfloor + 2$ and $0 < t \leq T$,

$$\begin{aligned} |Nu_N(t)| &\leq N \|u(0)\|_{\ell^2(\mathbb{Z})} \left(\sum_{n < \frac{N}{2}} |(e^{itL} \delta_N)_n|^2 \right)^{\frac{1}{2}} + N \|e^{itL} \delta_N\|_{\ell^2(\mathbb{Z})} \left(\sum_{n \geq \frac{N}{2}} |u_n(0)|^2 \right)^{\frac{1}{2}} \\ &\leq 2 \|u(0)\|_{\ell^2(\mathbb{Z})} \left(\sum_{n < \frac{N}{2}} (n - N)^2 |(e^{itL} \delta_N)_n|^2 \right)^{\frac{1}{2}} + 2 \left(\sum_{n \geq \frac{N}{2}} n^2 |u_n(0)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since $u(0) \in \mathcal{W}^2(\mathbb{Z})$, we have

$$(5.26) \quad \lim_{N \rightarrow \infty} \left(\sum_{n \geq \frac{N}{2}} n^2 |u_n(0)|^2 \right)^{\frac{1}{2}} = 0.$$

Similar to (5.25), we have

$$\begin{aligned} \sum_{n < \frac{N}{2}} (n - N)^2 |(e^{itL} \delta_N)_n|^2 &= \sum_{n+N < \frac{N}{2}} n^2 |(e^{itL} \delta_N)_{n+N}|^2 \\ &= \sum_{n < -\frac{N}{2}} n^2 |(S^{-N} e^{itL} S^N \delta_0)_n|^2 \\ &\leq \left(\sum_{n < -\frac{N}{2}} \frac{1}{n^2} \right) \left(\sum_{n \in \mathbb{Z}} n^4 |(S^{-N} e^{itL} S^N \delta_0)_n|^2 \right). \end{aligned}$$

Applying Theorem 2.22 in [14] with $p = 4$ and $\eta = \frac{5}{4}$, we get

$$\sum_{n < \frac{N}{2}} (n - N)^2 |(e^{itL} \delta_N)_n|^2 \lesssim N^{-2} (t^5 + 1) \ll T^{-5}.$$

Combining with (5.26), we finish the proof. \square

For any $n \in \mathbb{Z}$, let $\langle n \rangle := \sqrt{n^2 + 1}$ and for $k \in \mathbb{Z}$, we denote

$$(5.27) \quad a_k := \sup_{n \in \mathbb{Z}} \frac{1}{\langle n+k \rangle \langle n \rangle} \left| \int_{\mathbb{R}} [(\partial_E \mathcal{K}_{n+k})(\partial_E \mathcal{K}_n) + (\partial_E \mathcal{J}_{n+k})(\partial_E \mathcal{J}_n)] \psi dE \right|.$$

By (5.18), we can easily see that $a_k < 10c_5^2$ for every $k \in \mathbb{Z}$. The following lemma relates the lower bound of $\langle u(t) \rangle_2$ to $\{a_k\}$.

LEMMA 14. *There exists $c_8 > 0$, such that for sufficiently large $T > 0$, we have*

$$\langle u(T) \rangle_2 \geq \frac{c_8 T}{(T^\eta + \sum_{T^\eta < |k| \leq T^{101}} a_k)^{\frac{1}{2}}} - c_8^{-1}$$

Proof. By direct calculations, we can see

$$\begin{aligned} &\|\partial_E G_T(T, \cdot)\|_{\mathcal{L}^2(d\mu)}^2 \\ &= \sum_{|m|, |n| \leq \lfloor T^{100} \rfloor + 1} \langle m \rangle \langle n \rangle u_m(T) \bar{u}_n(T) \int_{\mathbb{R}} \frac{1}{\langle m \rangle \langle n \rangle} [(\partial_E \mathcal{K}_m)(\partial_E \mathcal{K}_n) + (\partial_E \mathcal{J}_m)(\partial_E \mathcal{J}_n)] \psi dE \\ &\leq \sum_{|m|, |n| \leq \lfloor T^{100} \rfloor + 1} \langle m \rangle \langle n \rangle |u_m(T)| |\bar{u}_n(T)| a_{m-n}. \end{aligned}$$

We can decompose $\|\partial_E G_T(T, \cdot)\|_{\mathcal{L}^2(d\mu)}^2$ into $I_1 + I_2$, corresponding to the summations $\sum_{\substack{|m|, |n| \leq \lfloor T^{100} \rfloor + 1 \\ |m-n| > T^\eta}}$

and $\sum_{\substack{|m|, |n| \leq \lfloor T^{100} \rfloor + 1 \\ |m-n| \leq T^\eta}}$ respectively. Since $a_k < 10c_5^2$ for every $k \in \mathbb{Z}$, we have

$$(5.28) \quad |I_2| \leq 10c_5^2 \sum_{|k| \leq T^\eta} \sum_{n \in \mathbb{Z}} \langle n+k \rangle \langle n \rangle |u_{n+k}(T)| |\bar{u}_n(T)| < 40c_5^2 T^\eta \langle u(T) \rangle_2^2,$$

and in view of Cauchy inequality, we obtain

$$(5.29) \quad |I_1| \leq \sum_{T^\eta < |k| \leq T^{101}} a_k \langle u(T) \rangle_2^2.$$

Then, by combining (5.28) and (5.29),

$$(5.30) \quad \left(40c_5^2 T^\eta + \sum_{T^\eta < |k| \leq T^{101}} a_k \right) \langle u(T) \rangle_2^2 \geq \|\partial_E G_T(T, E)\|_{\mathcal{L}^2(d\mu)}^2.$$

Similarly, we can see that

$$(5.31) \quad \left(40c_5^2 T^\eta + \sum_{T^\eta < |k| \leq T^{101}} a_k \right) \langle u(0) \rangle_2^2 \geq \|\partial_E G_T(0, E)\|_{\mathcal{L}^2(d\mu)}^2.$$

In view of (5.14), we have that $\lim_{T \rightarrow \infty} \|G_T(0, E)\|_{\mathcal{L}^2(d\mu)} = \|G(0, E)\|_{\mathcal{L}^2(d\mu)} > 0$. So, combining (5.23), (5.31), Lemma 12 and 13, we can find a constant $c'_8 > 0$ such that for sufficiently large $T > 0$,

$$\|\partial_E G_T(T, E)\|_{\mathcal{L}^2(d\mu)} \geq c'_8 T - \left(40c_4^2 T^\eta + \sum_{T^\eta < |k| \leq T^{101}} a_k \right)^{\frac{1}{2}} \langle u(0) \rangle_2.$$

The lemma follows together with (5.30). \square

Given $l \in \mathbb{Z}_+$ large enough, let $\mathcal{M}_l := \left[Q_l^{8\tau_0}, \min \left(Q_{l+1}^{\frac{\tau_1}{16}}, \overline{Q}_{l+1}^{\frac{\nu}{16}} \right) \right]$. The following lemma shows that $\{a_k\}_k$ is summable on the sequence of intervals $\{\mathcal{M}_l\}$.

LEMMA 15. *For sufficiently large l , for any $k \in \mathbb{Z} \setminus \{0\}$ such that $|k| \in \mathcal{M}_l$, we have $a_k \leq |k|^{-\frac{3}{2}}$.*

Proof. Recall the expressions of $\partial_E \mathcal{K}_n$ and $\partial_E \mathcal{J}_n$ in (5.15). Then for fixed $m, n \in \mathbb{Z} \setminus \{0\}$ with $|m - n| \in \mathcal{M}_l$, we have $\frac{1}{\langle m \rangle \langle n \rangle} \int_{\mathbb{R}} [(\partial_E \mathcal{K}_m)(\partial_E \mathcal{K}_n) + (\partial_E \mathcal{J}_m)(\partial_E \mathcal{J}_n)] \psi dE = \mathcal{P} + \mathcal{T}$, with

$$\begin{aligned} \mathcal{P} &:= \frac{1}{\langle m \rangle \langle n \rangle} \int_{\mathbb{R}} (\mathcal{B}_m \mathcal{B}_n + \mathcal{C}_m \mathcal{C}_n) \cos(\phi^{[m]} - \phi^{[n]}) \cdot \psi dE, \\ \mathcal{T} &:= \frac{1}{\langle m \rangle \langle n \rangle} \int_{\mathbb{R}} (\mathcal{B}_m \mathcal{C}_n - \mathcal{C}_m \mathcal{B}_n) \sin(\phi^{[m]} - \phi^{[n]}) \cdot \psi dE. \end{aligned}$$

We are going to show that $|\mathcal{P}|, |\mathcal{T}| \leq \frac{1}{2}|m - n|^{-\frac{3}{2}}$. We will detail the estimate of \mathcal{P} , and one can estimate \mathcal{T} in a similar way.

Noting that $\phi^{[m]}(\theta) - \phi^{[n]}(\theta) = \phi^{[m-n]}(\theta + n\alpha)$, we have

$$\mathcal{P} = \int_{\mathbb{R}} h(E, \theta) \psi(E) \cos \phi^{[m-n]}(E, \theta + n\alpha) dE,$$

with $h := \frac{1}{\langle m \rangle \langle n \rangle} (\mathcal{B}_m \mathcal{B}_n + \mathcal{C}_m \mathcal{C}_n)$. We can consider the approximated integral

$$\mathcal{P}_l = \int_{\mathbb{R}} h_l(E, \theta) \psi_l(E) \cos \phi_l^{[m-n]}(E, \theta + n\alpha) dE$$

instead, where $h_l := \frac{1}{\langle m \rangle \langle n \rangle} (\mathcal{B}_m^{(l)} \mathcal{B}_n^{(l)} + \mathcal{C}_m^{(l)} \mathcal{C}_n^{(l)})$. Indeed, in view of (5.16), (5.17), we have $|h_l -$

$h|_{J_k \cap \text{supp}(\psi)} \leq 4c_5^2 e^{-\overline{Q}_l^{D_2}}$ and

$$(5.32) \quad |h|_{J_k \cap \text{supp}(\psi)}, |h_l|_{C^2(J_k \cap \overline{Q}_l)} \leq 2c_5^2.$$

By Lemma 10, $|\phi_l^{[m-n]} - \phi^{[m-n]}|_{J_k \cap \bar{\Omega}} < c_2 |m-n| e^{-\bar{Q}_l^{D_2}}$. Since $|m-n| \in \mathcal{M}_l$ with \mathcal{M}_l defined by the $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible subsequence (Q_l) , we can see that, for large l ,

$$|m-n| e^{-\bar{Q}_l^{D_2}} \leq Q_{l+1}^{\frac{\tau_1}{16}} e^{-Q_{l+1}^{D_2 A^{-21}}} \leq Q_{l+1}^{-\tau_1} \leq |m-n|^{-5}.$$

And then

$$\left| h_l(E, \theta) \cos \phi_l^{[m-n]}(E, \theta + n\alpha) - h(E, \theta) \cos \phi^{[m-n]}(E, \theta + n\alpha) \right|_{J_k \cap \text{supp}(\psi)} < |m-n|^{-4}.$$

Combing with Proposition 3, we can see that

$$|\mathcal{P} - \mathcal{P}_l| \leq 5|m-n|^{-4} + 2c_5^2 |\psi - \psi_l|_{L^1(\mathbb{R})} \leq |m-n|^{-2}.$$

To compute \mathcal{P}_l , we apply the integration by parts on each J_k . Recalling the construction of C^∞ function χ , we have $h_l = \partial_E h_l = 0$ at both boundary points of J_k . So we have

$$\begin{aligned} & \int_{J_k} h_l(E, \theta) \psi_l(E) \cos \phi_l^{[m-n]}(E, \theta + n\alpha) dE \\ &= - \int_{J_k} \partial_E \left(\frac{h_l(E, \theta) \psi_l(E)}{\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha)} \right) \sin \phi_l^{[m-n]}(E, \theta + n\alpha) dE \\ (5.33) \quad &= - \int_{J_k} \frac{\partial_E (h_l(E, \theta) \psi_l(E))}{\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha)} \sin \phi_l^{[m-n]}(E, \theta + n\alpha) dE \end{aligned}$$

$$(5.34) \quad + \int_{J_k} \frac{h_l(E, \theta) \psi_l(E) \cdot \partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^2} \sin \phi_l^{[m-n]}(E, \theta + n\alpha) dE.$$

For the integrals in (5.33) and (5.34), by applying again the integration by parts, and noting that $\partial_E h_l = 0$ at each edge point of J_k , we have

$$\begin{aligned} & \int_{J_k} \frac{\partial_E (h_l(E, \theta) \psi_l(E))}{\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha)} \sin \phi_l^{[m-n]}(E, \theta + n\alpha) dE \\ &= \int_{J_k} \partial_E \left(\frac{\partial_E (h_l(E, \theta) \psi_l(E))}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^2} \right) \cos \phi_l^{[m-n]}(E, \theta + n\alpha) dE, \\ & \int_{J_k} \frac{h_l(E, \theta) \psi_l(E) \cdot \partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^2} \sin \phi_l^{[m-n]}(E, \theta + n\alpha) dE \\ &= - \int_{J_k} \partial_E \left(\frac{h_l(E, \theta) \psi_l(E) \cdot \partial_E \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^3} \right) \cos \phi_l^{[m-n]}(E, \theta + n\alpha) dE. \end{aligned}$$

By direct calculations using that $|\psi_l|_{C^2(\overline{\Omega}_l)} \lesssim l^2 Q_l^{\tau_1}$, and the estimates of h_l (see (5.32)), $\phi_l^{[m-n]}$ (see (5.6) and (5.7)), we can find a constant $c_9 > 0$ such that

$$\begin{aligned}
& \left| \partial_E \left(\frac{\partial_E(h_l(E, \theta)\psi_l(E))}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^2} \right) \right| \\
&= \left| \frac{\partial_E^2(h_l(E, \theta)\psi_l(E))}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^2} - \frac{2\partial_E(h_l(E, \theta)\psi_l(E)) \cdot (\partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha))}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^3} \right| \\
&\leq c_9 l^2 Q_l^{\tau_1} |m - n|^{-2}, \\
& \left| \partial_E \left(\frac{h_l(E, \theta)\psi_l(E) \cdot \partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^3} \right) \right| \\
&= \left| \frac{\partial_E(h_l(E, \theta)\psi_l(E)) \cdot \partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^3} + \frac{h_l(E, \theta)\psi_l(E) \cdot \partial_E^3 \phi_l^{[m-n]}(E, \theta + n\alpha)}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^3} \right. \\
&\quad \left. - \frac{3h_l(E, \theta)\psi_l(E) \cdot (\partial_E^2 \phi_l^{[m-n]}(E, \theta + n\alpha))^2}{(\partial_E \phi_l^{[m-n]}(E, \theta + n\alpha))^4} \right| \\
&\leq c_9 l^2 Q_l^{\tau_1} |m - n|^{-2}.
\end{aligned}$$

By $\tau_0 \geq \tau_1$ and $|m - n| \in \mathcal{M}_l$, both expressions above can be bounded by $\frac{1}{10}|m - n|^{-\frac{3}{2}}$ for sufficiently large l . This concludes that $|\mathcal{P}| \leq \frac{1}{2}|m - n|^{-\frac{3}{2}}$. \square

As a result of Lemma 15, we have $\sum_l \sum_{|k| \in \mathcal{M}_l} a_k < \infty$. Combined with Lemma 14, we immediately obtain the following.

COROLLARY A. *There exists $c_{10} > 0$, such that for any sufficiently large $l \geq 0$, for any $T > 0$ such that $[T^\eta, T^{101}] \subset \mathcal{M}_l$, we have $\langle u(T) \rangle_2 \geq c_{10} T^{1-\eta}$.*

In a similar way, with (Q_l) replaced by (R_l) , we can use \tilde{Z} , $\tilde{\phi}$ and $\tilde{\psi}$ (see Remark 6 and 7) to define truncated spectral transformation for the operator L in the same way. Similarly, we define $\tilde{\mathcal{M}}_l := \left[R_l^{8\tilde{\tau}_0}, \min(R_{l+1}^{\tau_1}, \overline{R}_{l+1}^{\frac{\nu}{16}}) \right]$. With the same proof, we have the analogue corollary.

COROLLARY B. *There exists $c_{11} > 0$, such that for any sufficiently large $l \geq 0$, for any $T > 0$ such that $[T^\eta, T^{101}] \subset \tilde{\mathcal{M}}_l$, we have $\langle u(T) \rangle_2 \geq c_{11} T^{1-\eta}$.*

Proof of Theorem 1. Without loss of generality, we assume that $\eta \in (0, 1)$. Recall that we have given τ_1 , τ_2 and \mathcal{A} in (5.3) and (5.4). Note that for any $l \geq 1$ we have $Q_{l+1}^{\tau_0} \geq R_l^{\mathcal{A}\tau_0} \geq R_l^{\tilde{\tau}_0}$ since $\mathcal{A} \geq \frac{\tilde{\tau}_0}{\tau_0}$. It is clear that $Q_l^{\tau_0} \leq R_l^{\tilde{\tau}_0}$ since $Q_l \leq R_l$ and $\tau_0 \leq \tilde{\tau}_0$. Thus for any sufficiently large T , we have two (nonexclusive) alternates :

- (1) There exists l such that $R_l^{8\tilde{\tau}_0} \leq T^\eta < Q_{l+1}^{8\tau_0}$. Then $T^{101} < Q_{l+1}^{808\tau_0/\eta} < \min(R_{l+1}^{\tau_2}, \overline{R}_{l+1}^{\frac{\nu}{16}})$ since $R_{l+1} \geq Q_{l+1}$, $\tau_2 > \frac{10^7\tau_0}{\eta}$ and $\overline{R}_{l+1}^{\frac{\nu}{16}} \geq Q_{l+1}^{\frac{\nu\mathcal{A}}{16}} > Q_{l+1}^{\frac{808\tau_0}{\eta}}$ for $\mathcal{A} > \frac{10^7\tau_0}{\eta\nu}$.
- (2) There exists l such that $Q_l^{8\tau_0} \leq T^\eta < R_l^{8\tilde{\tau}_0}$. Then $T^{101} < R_l^{808\tilde{\tau}_0/\eta} < \min(Q_{l+1}^{\tau_1}, \overline{Q}_{l+1}^{\frac{\nu}{16}})$ since $R_l^{\frac{808\tilde{\tau}_0}{\eta}} < Q_{l+1}^{\frac{808\tilde{\tau}_0}{\eta\mathcal{A}}} < \min(Q_{l+1}^{\tau_1}, \overline{Q}_{l+1}^{\frac{\nu}{16}})$. The last inequality follows from $\mathcal{A} > \frac{10^7\tilde{\tau}_0}{\eta} \max(\tau_1^{-1}, \nu^{-1})$ and $\overline{Q}_{l+1} > Q_{l+1}$.

Thus for any sufficiently large T , there exists $l \geq 0$ such that $[T^\eta, T^{101}]$ is contained in either \mathcal{M}_l or $\tilde{\mathcal{M}}_l$. The theorem follows from Corollary A and Corollary B. \square

5.6. Ballistic lower bound for $p \geq 0$. Till now we already have the lower bound for $p = 2$ and $p = 0$ (which is well-known as the ℓ^2 -conservation). Now we consider the general $p \geq 0$. We only have to show the lower bound

$$\lim_{t \rightarrow \infty} \frac{\langle u(t) \rangle_p^2}{t^{p-\eta}} = 0, \quad \forall \eta > 0, \quad \forall u(0) \in \mathcal{W}^p(\mathbb{Z}) \setminus \{0\}$$

for any even integer p . Indeed, by Cauchy inequality, we have

$$\langle u \rangle_{p_2} \lesssim \langle u \rangle_{p_3}^{\frac{p_1-p_2}{p_1-p_3}} \langle u \rangle_{p_1}^{\frac{p_2-p_3}{p_1-p_3}} \quad \text{for any } p_1 \geq p_2 \geq p_3 \geq 0.$$

- If $p > 2$, we can apply the above inequality for $p_1 = p$, $p_2 = 2\lfloor p/2 \rfloor$ and $p_3 = 0$, with the lower bound of $\langle u(t) \rangle_{2\lfloor p/2 \rfloor}$ and the upper bound of $\langle u(t) \rangle_0$ known. Then we get the lower bound in Theorem 1 for $\langle u(t) \rangle_p$.
- If $0 < p < 2$, we can apply the above inequality for $p_1 = 4$, $p_2 = 2$ and $p_3 = p$, with the lower bound of $\langle u(t) \rangle_2$ and the upper bound of $\langle u(t) \rangle_4$ known. Then we get the lower bound in Theorem 1 for $u(0) \in \mathcal{W}^4(\mathbb{Z}) \setminus \{0\}$.

For the p^{th} moment bound, with p an even integer, we need to make the following modifications to the proof presented above :

- (1) Let $r = p/2 + 2$. We consider $G_{Tr}(t, E)$ and $\partial_E^{r-2} G_{Tr}(t, E)$ instead of $G_T(t, E)$ and $\partial_E G_T(t, E)$.
- (2) Instead of Lemma 12, Lemma 13, we prove that $\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \|\partial_E^j F_{Tr}(t, E)\|_{\mathcal{L}^2(d\mu)} = 0$ for each $0 \leq j \leq r - 2$.
- (3) We consider $a_{k,r} := \sup_{n \in \mathbb{Z}} \frac{1}{\langle n+k \rangle^{r-2} \langle n \rangle^{r-2}} \left| \int_{\mathbb{R}} [(\partial_E^{r-2} \mathcal{K}_{n+k})(\partial_E^{r-2} \mathcal{K}_n) + (\partial_E^{r-2} \mathcal{J}_{n+k})(\partial_E^{r-2} \mathcal{J}_n)] \psi dE \right|$ instead of a_k in (5.27), with $\partial_E^{r-2} \mathcal{K}_n, \partial_E^{r-2} \mathcal{J}_n$ given in (5.19).
- (4) When verifying the summability of $a_{k,r}$ for $|k| \in \mathcal{M}_l$, we replace the estimates (5.16) – (5.18) by (5.20) – (5.22).

APPENDIX A. PROOF OF LEMMA 6 AND 7

In this section, the integer r is fixed. The implicit constants in the notations “ \lesssim ” and “ \gtrsim ” are allowed to depend on r .

Proof of Lemma 6

We follow the proof of Proposition 4.1 (1) in [6] and construct the conjugation cocycle B using iterations of algebraic conjugation, given by the following Lemma 16.

LEMMA 16. *For every $D > 0$, there exist $C_0 = C_0(D)$, $\epsilon_1 = \epsilon_1(D) > 0$ such that the following is true. Given $\epsilon \in (0, \epsilon_1)$, J_0 an open interval, let $\mathcal{U} \subset SL(2, \mathbb{C}) \times \mathbb{C}$ be the set of all (A, θ) such that $|A|_{J_0} < D$, $|R_\theta^{-1} A - id|_{J_0} < \epsilon$ and $\varrho = \min(\frac{1}{4}, |(R_{2\theta} - id)^{-1}|_{J_0}^{-1}) > \epsilon^{\frac{1}{r+4}}$, and let $E \mapsto (A(E), \theta(E))$ be a function in $C^\omega(J_0, \mathcal{U})$. There exists a real symmetric holomorphic function $F : \mathcal{U} \rightarrow SL(2, \mathbb{C})$ such that, with*

$$H := 1 + \max_{j=1, \dots, r} \left(|\partial_E^j \theta|_{J_0}^{\frac{r}{r-1}}, |\partial_E^j (R_\theta^{-1} A)|_{J_0}^{\frac{r}{r-1}} \epsilon^{-\frac{r}{r-1}} \right),$$

$B(E) = F(A(E), \theta(E))$ satisfies $BAB^{-1} = R_{\theta'}$ for some $\theta' \in C^\omega(J_0, \mathbb{C})$, and

$$|\partial_E^j (B - id)|_{J_0} < C_0 \epsilon \varrho^{-j-1} H^{\frac{1}{r}}, \quad |\partial_E^j (\theta - \theta')|_{J_0} < C_0 \epsilon H^{\frac{1}{r}}, \quad j = 0, \dots, r.$$

Proof. In this proof, for each function depending on $E \in J_0$, the norm we are considering is always $|\cdot|_{J_0}$. For convenience, we will not present the subscript “ J_0 ”.

Let $A^{(0)} := A$ and $\theta^{(0)} := \theta$. Suppose that $A^{(n)}$ and $\theta^{(n)}$ are already defined. Let $v^{(n)} \in sl(2, \mathbb{C})$ be small (thus unique) and satisfy that $A^{(n)} = e^{v^{(n)}} R_{\theta^{(n)}}$, with $v^{(n)} = \begin{pmatrix} x^{(n)} & y^{(n)} - 2\pi z^{(n)} \\ y^{(n)} + 2\pi z^{(n)} & -x^{(n)} \end{pmatrix}$. We define $\begin{pmatrix} \tilde{x}^{(n)} \\ \tilde{y}^{(n)} \end{pmatrix} := (R_{2\theta^{(n)}} - id)^{-1} \begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix}$ and $w^{(n)} := \begin{pmatrix} -\tilde{x}^{(n)} & -\tilde{y}^{(n)} \\ -\tilde{y}^{(n)} & \tilde{x}^{(n)} \end{pmatrix}$. Let $A^{(n+1)} = (e^{w^{(n)}})^{-1} A^{(n)} e^{w^{(n)}}$ and $\theta^{(n+1)} = \theta^{(n)} + z^{(n)}$. Then we have

$$A^{(n+1)} = (e^{w^{(n)}})^{-1} e^{v^{(n)}} R_{\theta^{(n)}} e^{w^{(n)}} = (e^{w^{(n)}})^{-1} e^{v^{(n)}} R_{\theta^{(n)}} e^{w^{(n)}} R_{-\theta^{(n+1)}} R_{\theta^{(n+1)}}.$$

Note that by our choice $R_{\theta^{(n)}} w^{(n)} R_{-\theta^{(n)}} = R_{2\theta^{(n)}} w^{(n)}$. So $v^{(n+1)}$ is defined by

$$e^{v^{(n+1)}} = (e^{w^{(n)}})^{-1} e^{v^{(n)}} R_{\theta^{(n)}} e^{w^{(n)}} R_{-\theta^{(n+1)}} = (e^{w^{(n)}})^{-1} e^{v^{(n)}} e^{R_{2\theta^{(n)}} w^{(n)}} R_{-z^{(n)}}.$$

Notice that we have

$$(A.1) \quad |\partial_E^j z^{(n)}| \lesssim |\partial_E^j v^{(n)}|, \quad \forall 0 \leq j \leq r.$$

We consider the k^{th} -order differential of $e^{v^{(n+1)}}$. It can be decomposed into $I_1 + I_2$, where

$$\begin{aligned} I_1 &= \partial_E^k ((e^{w^{(n)}})^{-1} e^{v^{(n)}} e^{R_{2\theta^{(n)}} w^{(n)}} R_{-z^{(n)}}) + (e^{w^{(n)}})^{-1} \partial_E^k e^{v^{(n)}} e^{R_{2\theta^{(n)}} w^{(n)}} R_{-z^{(n)}} \\ &\quad + (e^{w^{(n)}})^{-1} e^{v^{(n)}} \partial_E^k e^{R_{2\theta^{(n)}} w^{(n)}} R_{-z^{(n)}} + (e^{w^{(n)}})^{-1} e^{v^{(n)}} e^{R_{2\theta^{(n)}} w^{(n)}} \partial_E^k R_{-z^{(n)}} \end{aligned}$$

and I_2 is the sum of terms involving at least two differentiated components. By the definition of $w^{(n)}$, we have the identity

$$(A.2) \quad v^{(n)} + (R_{2\theta^{(n)}} - id)w^{(n)} - z^{(n)} \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} = 0.$$

By applying (A.2), we obtain $|I_1| \lesssim (|\partial_E^k v^{(n)}| + |\partial_E^k w^{(n)}|)(|v^{(n)}| + |w^{(n)}|)$.

By the definition of I_2 , we also have $|I_2| \lesssim (|\partial_E^k v^{(n)}| + |\partial_E^k w^{(n)}|)(|v^{(n)}| + |w^{(n)}|)$.

We will prove inductively that for all $k \geq 0$,

$$(A.3) \quad |(R_{2\theta^{(k)}} - id)^{-1}| \leq (2 - 2^{-k})\varrho^{-1}, \quad |\partial_E^j \theta^{(k)}| \leq (2 - 2^{-k})H^{\frac{j}{r}}, \quad j = 1, \dots, r$$

$$(A.4) \quad |\partial_E^j v^{(k)}| \lesssim \varrho^k \epsilon H^{\frac{j}{r}}, \quad j = 0, \dots, r.$$

$$(A.5) \quad |\partial_E^j w^{(k)}| \lesssim \varrho^{k-j-1} \epsilon H^{\frac{j}{r}}, \quad j = 0, \dots, r.$$

It is direct to verify (A.3) and (A.4) for $k = 0$ by the conditions in the lemma. Given (A.3), (A.4) for $0 \leq k \leq n$, it is easy to show by induction that for all $0 \leq k \leq n, 0 \leq j \leq r$

$$|\partial_E^j ((R_{2\theta^{(k)}} - id)^{-1})| \lesssim \varrho^{-j-1} H^{\frac{j}{r}}$$

Combining the above estimate with (A.1) and (A.2), we can deduce (A.5). In particular, we have verified (A.5) for $k = 0$.

Now assume that (A.3), (A.4), (A.5) are true for $0 \leq k \leq n$. For each $0 \leq j \leq r$, we decompose $\partial_E^j (e^{v^{(n+1)}})$ into I_1, I_2 as described above. By (A.4), (A.5) for $k = n$, we obtain

$$|I_1|, |I_2| \lesssim \varrho^{2n-r-2} \epsilon^2 H^{\frac{j}{r}}.$$

By $\epsilon \ll q^{r+3}$, we have $|\partial_E^j v^{(n+1)}| < q^{n+1} \epsilon H^{\frac{j}{r}}$. This gives us (A.4) for $k = n + 1$. By (A.1), (A.4) for n , and the fact that $\theta^{(n+1)} - \theta^{(n)} = z^{(n)}$, we see that for sufficiently small ϵ , we have (A.3) for $k = n + 1$. We deduce (A.5) from (A.3), (A.4). This completes the induction.

We define $B = \lim_{n \rightarrow \infty} (e^{w^{(n)}})^{-1} \dots (e^{w^{(1)}})^{-1} (e^{w^{(0)}})^{-1}$. It is clear that $BAB^{-1} = R_{\theta'}$ where $\theta' = \lim_{n \rightarrow \infty} \theta^{(n)}$. Recall that $|\partial_E^j (\theta^{(n)} - \theta^{(n+1)})| \lesssim |\partial_E^j v^{(n+1)}|$, then there exists $C_0 = C_0(D) > 0$ such that

$$|B - id| < C_0 q^{-1} \epsilon, \quad |\theta - \theta'| < C_0 \epsilon,$$

and for $j = 1, \dots, r$, we have

$$|\partial_E^j B| < C_0 q^{-j-1} \epsilon H^{\frac{j}{r}}, \quad |\partial_E^j (\theta - \theta')| < C_0 \epsilon H^{\frac{j}{r}}.$$

□

We will construct B in Lemma 6 inductively, by following the proof of Proposition 4.1 in [6]. Let $A_{(0)} = \bar{A}$, $\varphi_{(0)} = \bar{\varphi}$, $\xi_{(0)} = \bar{\xi} = R_{\bar{\varphi}}^{-1} \bar{A} - id$. Let $C_2 > 10$, $C_3 = C_0(10D)$ as in Lemma 16, $C_1 \gg rC_2C_3^{10r}$, $N = \frac{\delta h q^{2r+1}}{C_1 |\bar{\alpha}|}$ and $h_j = e^{-\delta \frac{j}{5N}} h$, $j \geq 0$. For $1 \leq i \leq N$ we define

- (1) $R_{\varphi_{(i)}} = B_{(i-1)} A_{(i-1)} B_{(i-1)}^{-1}$ where $B_{(i-1)}$ is obtained by applying Lemma 16 to $(A_{(i-1)}, \varphi_{(i-1)})$;
- (2) $A_{(i)}(E, x) = B_{(i-1)}(E, x + \bar{\alpha}) B_{(i-1)}^{-1} R_{\varphi_{(i)}(E, x)}$;
- (3) $\xi_{(i)} = R_{\varphi_{(i)}}^{-1} A_{(i)} - id$.

The following estimates are known (Claim 4.3 in [6]), thus we omit the proof.

LEMMA 17. *If ϵ_0 is sufficiently small, then*

- (1) $|B_{(i)} - id|_{h_i} < C_3 q^{-1} |\xi_{(i)}|_{h_i}$ for all $0 \leq i \leq N - 1$;
- (2) $|R_{\varphi_{(i)}}|_{h_i} < 2D$ for all $0 \leq i \leq N$;
- (3) $|(R_{2\varphi_{(i)}} - id)^{-1}|_{h_i} < 2q^{-1}$ for all $0 \leq i \leq N$;
- (4) $|\xi_{(i)}|_{h_i} \leq C_2^{-1} |\xi_{(i-1)}|_{h_{i-1}} \leq C_2^{-i} \epsilon_0$ for all $1 \leq i \leq N$.

Denote $\epsilon_i = C_2^{-i} \epsilon_0$ for $1 \leq i \leq N$. We will prove inductively that for $0 \leq i \leq N$,

$$(A.6) \quad 1 + \max_{1 \leq j \leq r} \left(|\partial_E^j \varphi_{(i)}|_{h_i}^{\frac{r}{j}}, |\partial_E^j \xi_{(i)}|_{h_i}^{\frac{r}{j}} \epsilon_i^{-\frac{r}{j}} \right) \leq (2 - 2^{-i}) H,$$

$$(A.7) \quad |\partial_E^j (B_{(i)} - id)|_{h_i} \leq C_3 \epsilon_i q^{-j-1} H^{\frac{j}{r}}, \quad j = 1, \dots, r.$$

For $i = 0$, (A.6) is given by the definition of H in Lemma 6. Apply Lemma 16 to $A_{(0)}$ and $\varphi_{(0)}$, we obtain (A.7). This verifies the induction hypothesis for $i = 0$.

Assume that we have (A.6), (A.7) for $0 \leq i \leq n - 1$. For each $1 \leq k \leq r$, we have

$$\begin{aligned} \partial_E^k \xi_{(n)} &= \sum_{1 \leq i_1 + i_2 \leq k} \partial_E^{i_1} (R_{-\varphi_{(n)}}) \partial_E^{k-i_1-i_2} (B_{(n-1)}(\cdot + \bar{\alpha}) B_{(n-1)}^{-1} - id) \partial_E^{i_2} (R_{\varphi_{(n)}}) \\ &\quad + R_{-\varphi_{(n)}} \partial_E^k (B_{(n-1)}(\cdot + \bar{\alpha}) B_{(n-1)}^{-1} - id) R_{\varphi_{(n)}}. \end{aligned}$$

Differentiate both sides of the identity $B_{(n-1)}(\cdot + \bar{\alpha}) B_{(n-1)}^{-1} - id = R_{\varphi_{(n)}} \xi_{(n)} R_{-\varphi_{(n)}}$, we can see

$$\begin{aligned} |\partial_E^k \xi_{(n)}|_{h_n} &\leq \sum_{1 \leq i_1 + i_2 + j_1 + j_2 \leq k} |\partial_E^{i_1} (R_{-\varphi_{(n)}})|_{h_n} |\partial_E^{j_1} (R_{\varphi_{(n)}})|_{h_n} |\partial_E^{k-i_1-i_2-j_1-j_2} \xi_{(n)}|_{h_n} |\partial_E^{i_2} (R_{-\varphi_{(n)}})|_{h_n} |\partial_E^{j_2} (R_{\varphi_{(n)}})|_{h_n} \\ (A.8) \quad &+ |\partial_E^k (B_{(n-1)}(\cdot + \bar{\alpha}) B_{(n-1)}^{-1} - id)|_{h_n}. \end{aligned}$$

By the analyticity, we have

$$\begin{aligned}
|\partial_E^k(B_{(n-1)}(\cdot + \bar{\alpha})B_{(n-1)}^{-1} - id)|_{h_n} &\leq \sum_{j=0}^k |\partial_E^j(B_{(n-1)}^{-1})|_{h_n} |\partial_E^{k-j}(B_{(n-1)}(\cdot + \bar{\alpha}) - B_{(n-1)})|_{h_n} \\
&\lesssim (h_{n-1} - h_n)^{-1} \bar{\alpha} \sum_{j=0}^k |\partial_E^j(B_{(n-1)}^{-1})|_{h_{n-1}} |\partial_E^{k-j}B_{(n-1)}|_{h_{n-1}} \\
&\lesssim \frac{N\bar{\alpha}}{\delta h} C_3^{k+1} \varrho^{-2k-1} \epsilon_{n-1} H^{\frac{k}{r}}.
\end{aligned}$$

The last step follows from (A.7) and the inductive estimate

$$|\partial_E^j(B_{(n-1)}^{-1})|_{h_{n-1}} \lesssim C_3^j \epsilon_{n-1} \varrho^{-2j} H^{\frac{j}{r}},$$

which we obtain from (A.7) by direct computation. Using (A.6) and (A.8), we can see that

$$|\partial_E^k \zeta_{(n)}|_{h_n} \lesssim \sum_{j=0}^{k-1} H^{\frac{k-j}{r}} |\partial_E^j \zeta_{(n)}|_{h_n} + C_3^{k+1} \frac{N\bar{\alpha}}{\delta h} \varrho^{-2k-1} \epsilon_{n-1} H^{\frac{k}{r}}.$$

Recall that $C_1 \gg r C_2 C_3^{10r}$ and $N = \frac{\delta h \varrho^{2r+1}}{C_1 |\bar{\alpha}|}$, by simple induction, we can see that

$$|\partial_E^k \zeta_{(n)}|_{h_n} \leq \epsilon_n H^{\frac{k}{r}}, \quad 0 \leq k \leq r.$$

By induction hypothesis (A.6) for $i = n-1$ and Lemma 17(1) and (3), we can apply Lemma 16 to $A_{(n-1)}$ and $\varphi_{(n-1)}$, and show that for some $C_4 = C_4(D)$,

$$|\partial_E^j(\varphi_{(n)} - \varphi_{n-1})|_{h_n} < C_4 \epsilon_{n-1} H^{\frac{j}{r}}.$$

Thus $|\partial_E^j \varphi_{(n)}|_{h_n} \leq [(2 - 2^{-n})H]^{\frac{j}{r}}$ for all $0 \leq j \leq r$ when ϵ_0 is sufficiently small. This gives us (A.6) for $i = n$. Combining with another application of Lemma 16 to $A_{(n)}$ and $\varphi_{(n)}$, we obtain (A.7) for $i = n$. This completes the induction.

We define $B = B_{(N-1)} \cdots B_{(0)}$, $\tilde{A} = A_{(N)}$, $\varphi = \varphi_{(N)}$. Then $R_\varphi \tilde{A} - id = \zeta_{(N)}$. It is direct to check (4.6) and (4.7) using (A.7) and Lemma 17. Since $\tilde{A} = R_\varphi + R_\varphi \zeta_{(N)}$, we have $\mathcal{Q}(\tilde{A}) = \mathcal{Q}(R_\varphi \zeta_{(N)})$. Then (4.8) follows from (A.6). \square

From now on, C_0 is a generic constant depending only on D, r that varies from line to line.

Proof of Lemma 7

The following lemma is Claim 4.4 of [6]. We present it here without proof.

LEMMA 18. *If $C_5(D) > 0$ is sufficiently large, and if ϵ_0 is sufficiently small and $|\bar{\alpha}| < C_5^{-1} \delta h \varrho^2$, then*

$$|(R_{\varphi(x+\alpha)+\varphi(x)} - id)^{-1}|_{e^{-\delta/3h}} \lesssim \varrho^{-1}.$$

Following [6], we define $\tilde{\zeta}, \tilde{\xi}$ by letting $R_{\tilde{\zeta}} = \frac{\tilde{G} - \mathcal{Q}(\tilde{G})}{\det(\tilde{G} - \mathcal{Q}(\tilde{G}))^{\frac{1}{2}}}$ and $\tilde{\xi} = R_{-\tilde{\zeta}} \tilde{G} - id$. Here \mathcal{Q} is defined in Lemma 6. Then

$$(A.9) \quad (\tilde{G} - \mathcal{Q}(\tilde{G}))\tilde{\xi} = (\det(\tilde{G} - \mathcal{Q}(\tilde{G}))^{\frac{1}{2}} - 1)\tilde{G} + \mathcal{Q}(\tilde{G}).$$

After differentiating (A.9), we obtain that, for each $k \geq 1$,

$$(A.10) \quad \sum_{j=0}^k \partial_E^{k-j}(\tilde{G} - \mathcal{Q}(\tilde{G})) \partial_E^j \tilde{\xi} = \sum_{j=0}^k \partial_E^j (\det(\tilde{G} - \mathcal{Q}(\tilde{G}))^{\frac{1}{2}} - 1) \partial_E^{k-j} \tilde{G} + \partial_E^k \mathcal{Q}(\tilde{G}).$$

Then, by a direct computation, we have, for any $k \geq 0$,

$$(A.11) \quad |\partial_E^k \mathcal{Q}(\tilde{G})|, |\partial_E^k(\tilde{G} - \mathcal{Q}(\tilde{G}))| \lesssim |\partial_E^k \tilde{G}|,$$

$$(A.12) \quad |\partial_E^k(\det(\tilde{G} - \mathcal{Q}(\tilde{G})) - 1)| \lesssim \sum_{j=0}^k |\partial_E^{k-j} \tilde{G}| |\partial_E^j \mathcal{Q}(\tilde{G})|,$$

where $|\cdot| = |\cdot|_{e^{-\frac{\delta}{2}h}}$. By choosing ϵ_0 small, we can assume that $\tilde{G} - \mathcal{Q}(\tilde{G})$ is invertible and $\frac{1}{2} < |\det(\tilde{G} - \mathcal{Q}(\tilde{G}))| < 2$. Thus

$$(A.13) \quad \begin{aligned} |\partial_E^k(\det(\tilde{G} - \mathcal{Q}(\tilde{G}))^{\frac{1}{2}} - 1)| &\lesssim \sum_l \sum_{\substack{i_j \geq 1 \\ i_1 + \dots + i_l = k}} \prod_{j=1}^l |\partial_E^{i_j} \det(\tilde{G} - \mathcal{Q}(\tilde{G}))| \\ &\lesssim \sum_l \sum_{\substack{i_j \geq 1, 0 \leq p_j \leq i_j \\ i_1 + \dots + i_l = k}} \prod_{j=1}^l |\partial_E^{i_j - p_j} \mathcal{Q}(\tilde{G})| |\partial_E^{p_j} \tilde{G}|. \end{aligned}$$

Combining with (A.10)–(A.13), we obtain

$$(A.14) \quad \begin{aligned} |\partial_E^k \tilde{\xi}| &\lesssim \sum_{i=0}^{k-1} |\partial_E^i \tilde{\xi}| |\partial_E^{k-i} \tilde{G}| + |\partial_E^k \tilde{G}| |\tilde{G}| |\mathcal{Q}(\tilde{G})| + |\partial_E^k \mathcal{Q}(\tilde{G})| \\ &\quad + \sum_{i=1}^k |\partial_E^{k-i} \tilde{G}| \sum_{j=1}^i \sum_{\substack{0 \leq p_m \leq j_m \\ \sum_{m=1}^l j_m = j \\ j_m \geq 1}} \prod_{m=1}^l |\partial_E^{j_m - p_m} \mathcal{Q}(\tilde{G})| |\partial_E^{p_m} \tilde{G}|. \end{aligned}$$

We have the equality $R_{-\zeta} B(\cdot + \alpha) R_{\zeta} (id + \tilde{\zeta}) B^{-1} = R_{\tilde{\zeta} - \zeta} (id + \tilde{\zeta})$. Thus

$$\begin{aligned} (R_{\tilde{\zeta} - \zeta} - id)(id + \tilde{\zeta}) &= R_{-\zeta} (B(\cdot + \alpha) - id) R_{\zeta} (id + \tilde{\zeta}) (id - B) B^{-1} \\ &\quad + (id + \tilde{\zeta}) (id - B) B^{-1} + R_{-\zeta} (B(\cdot + \alpha) - id) R_{\zeta} (id + \tilde{\zeta}). \end{aligned}$$

After differentiating both side, we obtain for each $k \geq 0$ that

$$(A.15) \quad \begin{aligned} |\partial_E^k(\tilde{\zeta} - \zeta)| &\lesssim \sum_{i=1}^k |\partial_E^{k-i}(\tilde{\zeta} - \zeta)| |\partial_E^i \tilde{\xi}| \\ &\quad + \sum_{\substack{i_1 + \dots + i_6 = k \\ i_j \geq 0}} |\partial_E^{i_1} R_{-\zeta}| |\partial_E^{i_3} R_{\zeta}| |\partial_E^{i_4} (id + \tilde{\zeta})| |\partial_E^{i_2} (B - id)| |\partial_E^{i_5} (B - id)| |\partial_E^{i_6} (B^{-1})|. \\ &\quad + \sum_{\substack{i_1 + i_2 + i_3 = k \\ i_j \geq 0}} |\partial_E^{i_1} (id + \tilde{\zeta})| |\partial_E^{i_2} (B - id)| |\partial_E^{i_3} (B^{-1})| \\ &\quad + \sum_{\substack{i_1 + i_2 + i_3 + i_4 = k \\ i_j \geq 0}} |\partial_E^{i_1} R_{-\zeta}| |\partial_E^{i_2} (B - id)| |\partial_E^{i_3} R_{\zeta}| |\partial_E^{i_4} (id + \tilde{\zeta})|. \end{aligned}$$

We need a good upper bound for $|\partial_E^k \mathcal{Q}(\tilde{G})|$. Note that for each $\theta \in \mathbb{C}$, any matrix M , we have

$$R_{-\theta} M = \mathcal{Q}(M R_{\theta}), \quad R_{\theta} \mathcal{Q}(M) = \mathcal{Q}(R_{\theta} M).$$

By the commutation relation between G and \tilde{A} , we obtain $\tilde{G}(x + \bar{\alpha})\tilde{\tilde{A}}(x) = \tilde{\tilde{A}}(x + \alpha)\tilde{G}(x)$. After taking the derivatives, we obtain by the commutation relation, that for each $k \geq 1$

$$\sum_{i=0}^k \partial_E^{k-i} \tilde{G}(x + \bar{\alpha}) \partial_E^i \tilde{\tilde{A}}(x) = \sum_{i=0}^k \partial_E^{k-i} \tilde{\tilde{A}}(x + \alpha) \partial_E^i \tilde{G}(x),$$

and

$$\begin{aligned} (A.16) \quad & (R_{-\varphi(x)} - R_{\varphi(x+\alpha)}) \mathcal{Q} \left(\partial_E^k \tilde{G}(x) \right) \\ &= \mathcal{Q} \left(\partial_E^k \tilde{G}(x) R_{\varphi(x)} - R_{\varphi(x+\alpha)} \partial_E^k \tilde{G}(x) \right) \\ &= \mathcal{Q} \left((\partial_E^k \tilde{G}(x) - \partial_E^k \tilde{G}(x + \bar{\alpha})) R_{\varphi(x)} \right) + \mathcal{Q} \left((\tilde{\tilde{A}}(x + \alpha) - R_{\varphi(x+\alpha)}) \partial_E^k \tilde{G}(x) \right) \\ &\quad + \mathcal{Q} \left(\partial_E^k \tilde{G}(x + \bar{\alpha}) (R_{\varphi(x)} - \tilde{\tilde{A}}(x)) \right) - \sum_{i=1}^k \mathcal{Q} \left((\partial_E^{k-i} \tilde{G}(x + \bar{\alpha}) - \partial_E^{k-i} R_{\tilde{\zeta}(x+\bar{\alpha})}) \partial_E^i \tilde{\tilde{A}}(x) \right) \\ &\quad - \sum_{i=1}^k \mathcal{Q} \left(\partial_E^{k-i} R_{\tilde{\zeta}(x+\bar{\alpha})} \partial_E^i \tilde{\tilde{A}}(x) \right) + \sum_{i=1}^k \mathcal{Q} \left(\partial_E^i \tilde{\tilde{A}}(x + \alpha) (\partial_E^{k-i} \tilde{G}(x) - \partial_E^{k-i} R_{\tilde{\zeta}(x)}) \right) \\ &\quad + \sum_{i=1}^k \mathcal{Q} \left(\partial_E^i \tilde{\tilde{A}}(x + \alpha) \partial_E^{k-i} R_{\tilde{\zeta}(x)} \right). \end{aligned}$$

To simplify notations, let $C_1 \gg r$, $N = \frac{\delta h \varrho^{2r+1}}{C_1 |\bar{\alpha}|}$, $g_n = e^{-\frac{\delta}{2} - \frac{n\delta}{2(r+1)N}} h$, and let $|\cdot|_n := |\cdot|_{g_n}$ in the following computations. For $j = 0, \dots, r$, we denote $P^{(j)}(x) := \mathcal{Q} \left(\partial_E^j \tilde{G}(x) \right)$ and

$$P_1^{(j)}(x) := P^{(j)}(x + \bar{\alpha}) - P^{(j)}(x), \quad P_2^{(j)}(x) := \mathcal{Q} \left(\partial_E^j \tilde{G}(x) R_{\varphi(x)} - R_{\varphi(x+\alpha)} \partial_E^j \tilde{G}(x) \right).$$

By hypothesis in Lemma 6, Lemma 7 and (4.6), (4.8) (Note that the conditions and conclusions of Lemma 6 holds for Lemma 7), we see that

$$(A.17) \quad |\partial_E^j \tilde{G}|, |\partial_E^j \tilde{\tilde{A}}| \lesssim H^{\frac{j}{r}}, \quad \forall j = 0, \dots, r,$$

$$(A.18) \quad |\partial_E^j \mathcal{Q}(\tilde{\tilde{A}})| < C_0 \epsilon_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0 |\bar{\alpha}|}} H^{\frac{j}{r}}, \quad \forall j = 1, \dots, r.$$

Assume that for $0 \leq k \leq r$, we have

$$(A.19) \quad |\partial_E^j \tilde{\zeta}|_{jN} < C_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0 |\bar{\alpha}|}} H^{\frac{j}{r}}, \quad |\partial_E^j (\tilde{\zeta} - \zeta)|_{jN} < C_0 H^{\frac{j}{r}} \epsilon_0, \quad 0 \leq j \leq k-1.$$

Notice that this hypothesis is empty for $k = 0$. For $kN \leq n \leq (k+1)N - 1$, by Lemma 18 and the analyticity, we obtain

$$(A.20) \quad |P^{(j)}|_{n+1} < C_0 \varrho^{-1} |P_2^{(j)}|_{n+1}, \quad |P_1^{(j)}|_{n+1} < \frac{C_0 |\bar{\alpha}|}{g_n - g_{n+1}} |P^{(j)}|_n.$$

By (A.16)–(A.19), we have

$$\begin{aligned} |P_2^{(k)}|_{n+1} &\lesssim |P_1^{(k)}|_{n+1} + |\partial_E^k \tilde{G}|_{n+1} |R_{\varphi} - \tilde{\tilde{A}}|_{n+1} \\ &\quad + \sum_{i=1}^k \left(|\partial_E^{k-i} (\tilde{G} - R_{\tilde{\zeta}})|_{n+1} |\partial_E^i \tilde{\tilde{A}}|_{n+1} + |\partial_E^{k-i} \tilde{\zeta}|_{n+1} |\partial_E^i \mathcal{Q}(\tilde{\tilde{A}})|_{n+1} \right) \\ &\lesssim |P_1^{(k)}|_{n+1} + C_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0 |\bar{\alpha}|}} H^{\frac{k}{r}}. \end{aligned}$$

Then by (A.20), we obtain $|P^{(k)}|_{n+1} \lesssim \frac{C_0(r+1)N|\tilde{\alpha}|}{\delta h} \varrho^{-1} |P^{(k)}|_n + C_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0|\tilde{\alpha}|}} H_r^{\frac{k}{r}}$. Using (A.17), (A.11) and the facts that $C_1 \gg r$, $N = \frac{\delta h \varrho^{2r+1}}{C_1|\tilde{\alpha}|}$, we get

$$(A.21) \quad |Q(\partial_E^k \tilde{G})|_{(k+1)N} = |P^{(k)}|_{(k+1)N} < C_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0|\tilde{\alpha}|}} H_r^{\frac{k}{r}}.$$

By (A.14) and (A.15), combining with (4.6), (A.17), (A.11), (A.21), we have

$$|\partial_E^k \tilde{\zeta}|_{(k+1)N} < C_0 e^{-\frac{\delta h \varrho^{2r+1}}{C_0|\tilde{\alpha}|}} H_r^{\frac{k}{r}}, \quad |\partial_E^k (\tilde{\zeta} - \zeta)|_{(k+1)N} < C_0 H_r^{\frac{k}{r}} \epsilon_0.$$

This completes the induction and thus completes the proof. \square

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